The Open University

MT365 Graphs, networks and design

Graphs 3

Planarity and colouring

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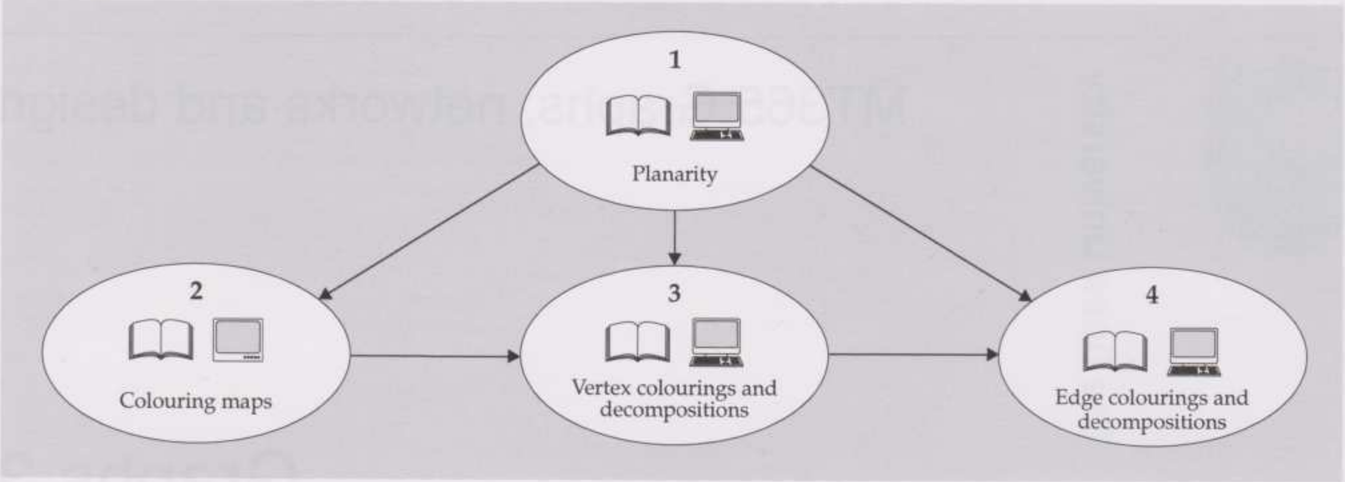
Study guide



The most important sections of this unit are Sections 1, 3 and 4, and you should make sure that you consolidate the main ideas of these sections. There are computer activities associated with these sections.

Section 2 is a television section; the programme can be watched at any time during your study of this unit.

Sections 3.3 and 4.3 tie together several ideas from the first three *Graphs* units; if you are short of time, you may prefer to skim through them, leaving detailed study of them until later.

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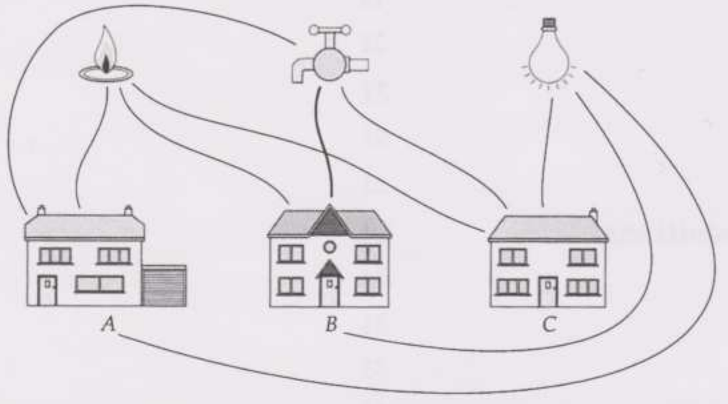
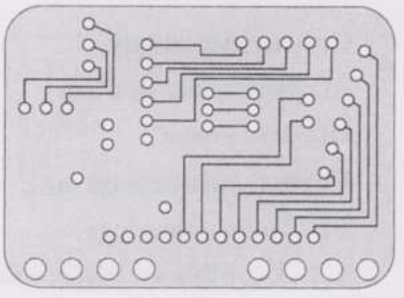
Contents

Introduction

In the *Introduction* unit, you met two problems that are related to the themes of this unit — the *utilities problem* and the *map colouring problem.*

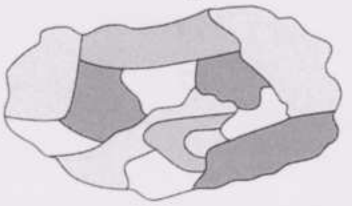
In the utilities problem, three neighbours are to be connected to the three utilities gas, water and electricity, in such a way that the connections do not cross. A more practical version of this type of problem arises in the design of printed circuit boards, on which electronic components are connected by means of conducting strips printed directly onto the flat board of insulating material. Such printed connections may not cross, since this would lead to undesirable electrical contact at crossing points.





In Section 1, *Planarity,* we generalize these ideas by considering the problem of determining whether a given graph can be drawn in the plane without any of its edges crossing, and we develop techniques for tackling problems such as the utilities problem. We introduce Euler's formula and discuss the duality of planar graphs — both of these topics provide a link with *Design 1.* We also describe a simple method for determining whether a given graph is planar.

In the map colouring problem, we wish to colour the countries of a map in such a way that any two countries with a common border are coloured differently. We saw in the *Introduction* unit that four colours are sufficient to colour the maps we met earlier, but left open the question as to whether four colours are enough to colour *all* maps.



Section 2, *Colouring maps,* is a television section on the map colouring problem. The television programme includes a discussion with Professors Kenneth Appel and Wolfgang Haken, who settled the above question in 1976 after it had remained unanswered for 124 years.

In Section 3, *Vertex colourings and decompositions,* we look at problems involving the colouring of the vertices of a graph. We also introduce an algorithm for colouring the vertices of a given graph. We then widen our scope and consider a range of problems that involve splitting the set of vertices of a graph into disjoint subsets with particular properties. Such problems include the vertex colouring problem.

Finally, in Section 4, *Edge colourings and decompositions,* we look at the corresponding problems involving the colouring of the edges of a graph. Our discussion involves problems relating to printed circuits and the scheduling of examinations.

The problems discussed in this unit vary considerably in terms of how much work has been done on them, and how much is known about their

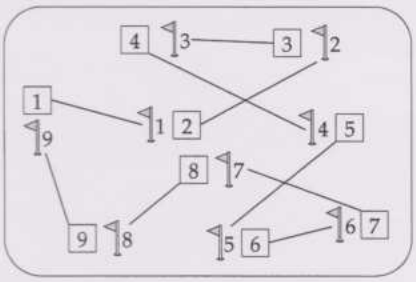
solutions. Some have elegant theoretical solutions, but lack efficient algorithms. Others have algorithms that work well in practice, but lack theoretical solutions. For some problems (such as that of determining whether a given graph is planar), there exist theoretical solutions and several efficient algorithms. And there is the map colouring problem, which has a theoretical solution that is so complicated that it originally took 1200 hours of computer time to find it!

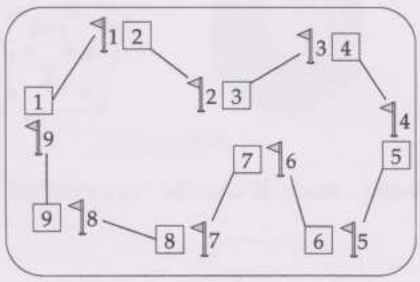
1 Planarity

In this section we investigate the properties of graphs that can be drawn in the plane without any of their edges crossing; such graphs are called *planar graphs*. In particular, we determine whether the complete bipartite graph $K_{3,3}$ is planar, thereby solving the utilities problem. We also discuss Euler's formula and Kuratowski's theorem; the latter is an important result which gives a necessary and sufficient condition for a graph to be planar. We conclude by describing a heuristic algorithm that can be used to determine whether a given graph is planar.

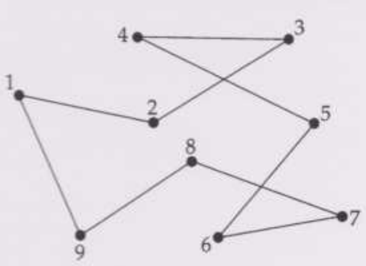
1.1 Examples of planar graphs

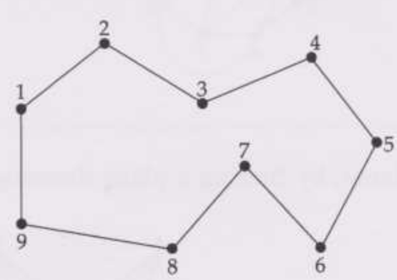
Suppose you wish to design a nine-hole golf course. It is advisable to do so in such a way that no two of the fairways intersect, as this would cause inconvenience and possible danger to the golfers. For example, the first diagram below would be unsuitable, whereas the second would be more appropriate.



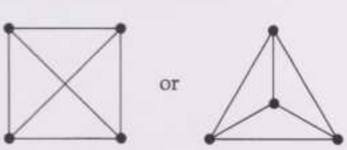


We can represent each of these layouts by the cycle graph C_9; the nine vertices correspond to the tees (or greens), and the edges correspond to the fairways. In the first drawing, some of the edges cross, whereas in the second drawing there are no crossings.

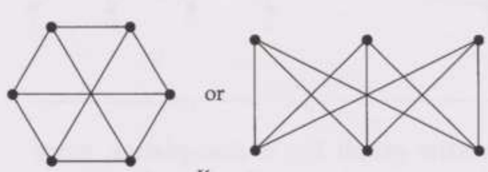




You have seen many instances of graphs drawn in more than one way. For example, the complete graph K_4 and the complete bipartite graph $K_{3,3}$ can be drawn as follows.



K_4



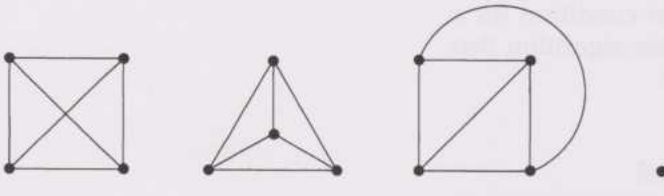
$K_{3,3}$

The particular drawing we choose often depends on the use to which the graph is to be put. As we have seen, it is sometimes useful to know

whether we can draw a graph in such a way that no two edges cross. For some graphs, such as K_4, it is possible to find a drawing that involves no 'crossings', whereas for others, such as $K_{3,3}$, there are no such drawings, as you will see. This leads us to formulate the following definitions.

> **Definitions**
>
> A graph G is **planar** if it can be drawn in the plane in such a way that no two edges meet except at a vertex with which they are both incident. Any such drawing is a **plane drawing** of G.
>
> If no plane drawing of G exists, then G is **non-planar**.

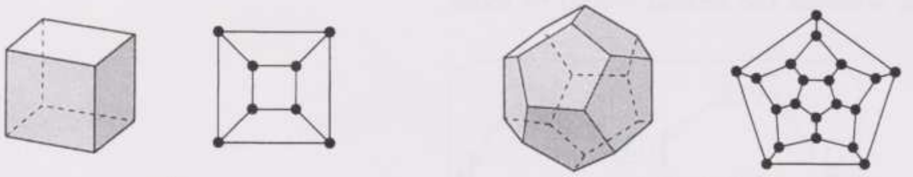
For example, the graph K_4 is planar, since it can be drawn in the plane without edges crossing. The following diagram shows three plane drawings of K_4.



K_4 plane drawings of K_4

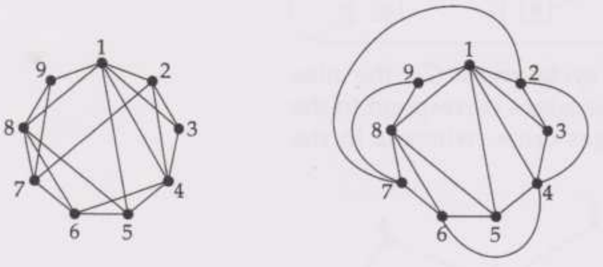
The graphs of the cube and dodecahedron are planar, since they can be drawn as follows.

These graphs were introduced in *Graphs 1*.



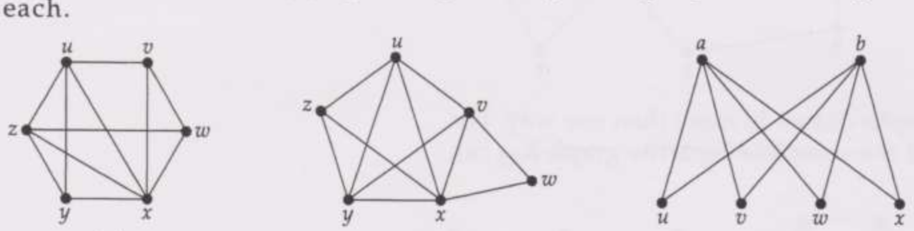
cube dodecahedron

Similarly, the following graph is planar, since it can be 'unravelled' as shown.



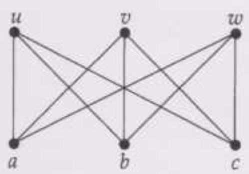
Problem 1.1

Show that the following graphs are planar, by finding a plane drawing of each.

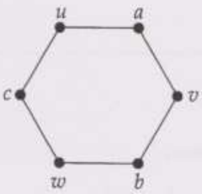


(a) (b) (c)

On the other hand, the complete bipartite graph $K_{3,3}$ is non-planar, since every drawing of it must contain at least one crossing. To see why this is, note that $K_{3,3}$ has a cycle of length 6 (*uavbwcu*). This must appear in any plane drawing as a hexagon, not necessarily regular.

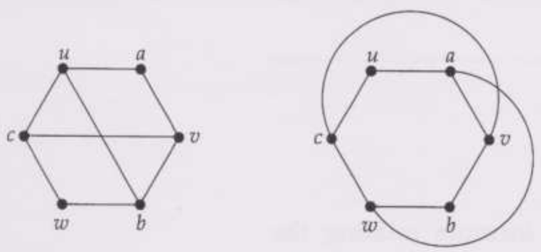


$K_{3,3}$



cycle in $K_{3,3}$

We must now insert the edges ub, vc and wa. Only one of them can be drawn *inside* the hexagon, since any two would cross. Similarly, only one of them can be drawn *outside* the hexagon, since any two would cross.



It is therefore impossible to insert all three of these edges without creating a crossing. This demonstrates that $K_{3,3}$ is non-planar.

Problem 1.2

Explain why the utilities problem has no solution — that is, why it is not possible to connect each of the three houses to the three utilities so that no connections cross.

Problem 1.3

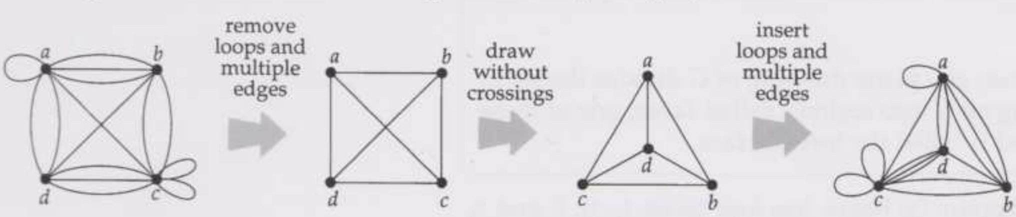
Give an explanation, similar to that given for $K_{3,3}$, to demonstrate that the complete graph K_5 is non-planar.

Problem 1.4

There was once a king with five sons. In his will he stated that after his death each son should build a castle, and that the five castles should be connected in pairs by non-intersecting roads. Can the terms of the will be satisfied?

This is a form of 'Möbius's problem', stated by August Möbius around 1840.

Note that, when studying planar graphs, we may restrict our attention to simple graphs whenever it is convenient to do so. If a planar graph has multiple edges or loops, we replace the multiple edges by a single edge and remove the loops. After drawing the resulting simple graph without crossings, we can then insert the loops and multiple edges, as follows.



Problem 1.5

Decide whether each of the following statements is TRUE or FALSE, and give a reason or counter-example as appropriate.

(a) Every subgraph of a planar graph is planar.

(b) Every subgraph of a non-planar graph is non-planar.

(c) Every graph that contains a planar subgraph is planar.

(d) Every graph that contains a non-planar subgraph is non-planar.

Problem 1.6

(a) Which trees are planar?

(b) For which values of n is the cycle graph C_n planar?

(c) For which values of n is the complete graph K_n planar?

(d) For which values of s are the complete bipartite graphs $K_{1,s}$ and $K_{2,s}$ planar?

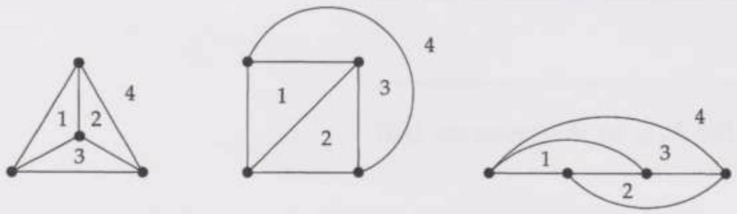
(e) For which values of r and s ($r \leq s$) is the complete bipartite graph $K_{r,s}$ planar?

1.2 Euler's formula

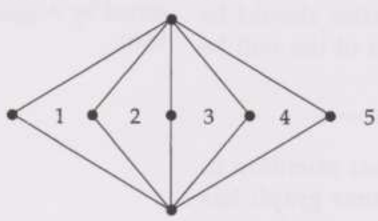
In *Design 1* we introduced Euler's polyhedron formula relating the numbers of vertices, edges and faces of a polyhedron. In this subsection we introduce an analogous formula for a plane drawing of a planar graph. First, we define a *face* of such a drawing.

Every plane drawing of a planar graph divides the plane into a number of regions. For example, any plane drawing of K_4 divides the plane into four regions — three triangles (3-cycles) and one 'infinite region'.

More precisely, the graph divides the points of the plane *not lying on the graph* into regions; the regions do not include the vertices and edges forming their boundaries.



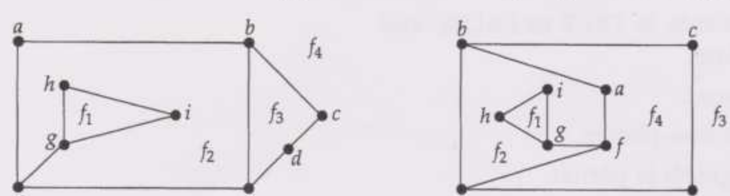
Similarly, any plane drawing of $K_{2,5}$ divides the plane into five regions — four quadrilaterals and one 'infinite region'.



We formalize these ideas as follows.

> **Definitions**
>
> Let G be a planar graph. Then any plane drawing of G divides the set of points of the plane not lying on G into regions, called **faces**; one of these faces is of infinite extent and is called the **infinite face**.

For example, the graph in diagram (a) below has four faces, f_1, f_2, f_3 and f_4, where f_4 is the infinite face. An alternative drawing of G, in which the faces have the same boundaries but f_3 is the infinite face, is given in diagram (b).



(a) (b)

Problem 1.7

Find plane drawings of the above graph in which:

(a) f_2 is the infinite face;

(b) f_1 is the infinite face.

We define the *degree* of each face of a plane drawing of a planar graph as follows.

Definitions

Let G be a planar graph, and let f be any face of a plane drawing of G. Then the **degree of** f, denoted by **deg** f, is the number of edges encountered in a walk around the boundary of the face f.

If all faces have the same degree g, then G is **face-regular of degree** g.

K_4

An example of a face-regular graph is K_4, which is face-regular of degree 3.

For example, in each drawing of the graph G in diagrams (a) and (b) above, we have

$\deg f_1 = 3$ and $\deg f_3 = 4$.

Note that both sides of the edge gf lie on the boundary of a single face f_2 of G, and so must be counted *twice* as we walk around the boundary of the face; thus, $\deg f_2 = 9$.

If we find the sum of all the face degrees we obtain $3 + 4 + 9 + 6 = 22$, which is exactly twice the number of edges of G. This makes us suspect that the handshaking lemma for graphs has a 'face analogue' for the faces in a plane drawing of a planar graph. This is indeed the case, and we refer to it as the *handshaking lemma for planar graphs*.

Theorem 1.1: handshaking lemma for planar graphs

In any plane drawing of a planar graph, the sum of all the face degrees is equal to twice the number of edges.

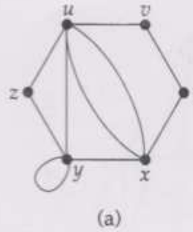
Recall from *Design 1* that there is an analogous handshaking lemma for polyhedra.

Proof

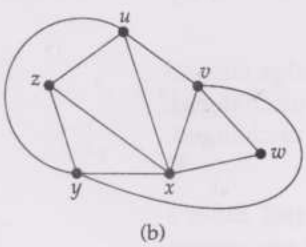
Since each edge has two sides (which may lie on the boundary of a single face or on the boundaries of two different faces), it must contribute exactly 2 to the sum of the face degrees. The result follows immediately. ■

Problem 1.8

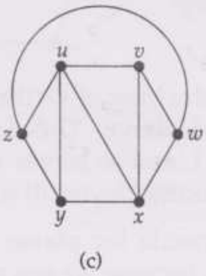
Verify the above version of the handshaking lemma for each of the following plane drawings of planar graphs.



(a)



(b)



(c)

Problem 1.9

For each of the plane drawings in Problem 1.8, count the numbers of vertices, edges and faces, and find the value of

(number of vertices) − (number of edges) + (number of faces).

In the solution to Problem 1.9, you saw that, for each of the plane drawings under consideration,

(number of vertices) – (number of edges) + (number of faces) = 2.

This equation holds for any plane drawing of a connected planar graph, and is known as *Euler's formula.*

> ### Theorem 1.2: Euler's formula for planar graphs
>
> Let G be a connected planar graph, and let n, m and f denote, respectively, the numbers of vertices, edges and faces in a plane drawing of G. Then
>
> $$n - m + f = 2.$$

Euler's formula tells us that each plane drawing of a given connected planar graph with n vertices and m edges must have the same number of faces — namely, $2 - n + m$.

Proof

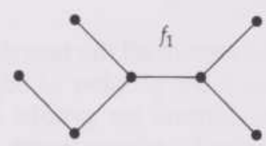
A plane drawing of any connected graph G can be constructed by taking a spanning tree and adding edges to it, one at a time, until a plane drawing of the graph G is obtained.

Graphs 2, Section 3.1.

We prove Euler's formula by showing that:

(a) for any spanning tree, $n - m + f = 2$;

(b) adding an edge does not change the value of $n - m + f$.

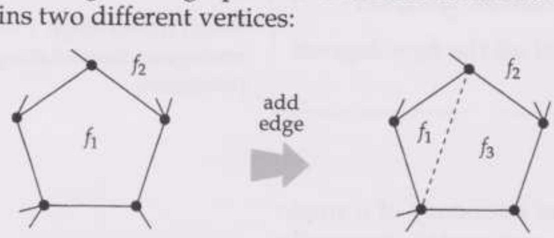


First, we prove statement (a). Let T be any spanning tree of G; then we may draw T in the plane without crossings. Since T has n vertices and $n - 1$ edges, and there is only 1 face (the infinite face), we have

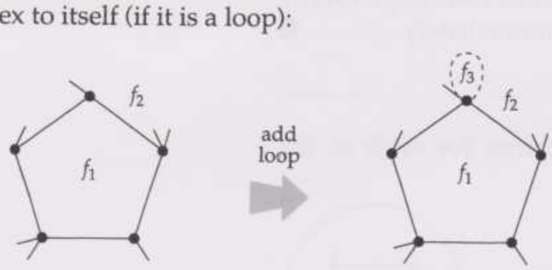
$$n - m + f = n - (n - 1) + 1 = 2,$$

as required.

Now we prove statement (b) by adding in the other edges one at a time until a plane drawing of the graph G is obtained. At each stage the added edge either joins two different vertices:

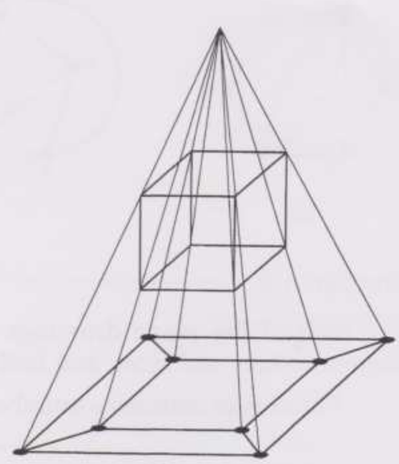


or joins a vertex to itself (if it is a loop):



In each case, since we have a plane drawing of G, the added edge cuts an existing face in two, as illustrated above. This leaves n unchanged, increases m by 1, and increases f by 1, and so leaves $n - m + f$ unchanged. Since $n - m + f = 2$ throughout the process, the result follows. ■

The connection between Euler's formula for planar graphs and Euler's formula for polyhedra is immediate, because we can represent any convex polyhedron as a planar graph by projecting it down onto a plane; this method of projection, called *stereographic projection*, does not alter the value of $n - m + f$, which is 2 in both situations.



stereographic projection of cube

Problem 1.10

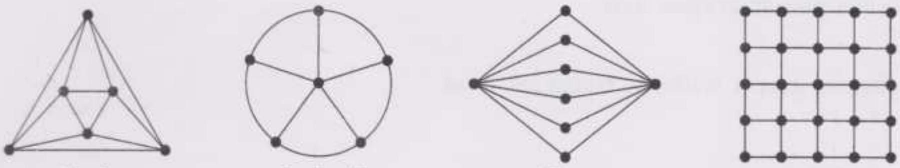
Verify Euler's formula for each of the following graphs:

(a) the octahedron graph;

(b) the 'wheel' with k spokes;

(c) the complete bipartite graph $K_{2,k}$;

(d) the graph formed from the vertices and edges of a $k \times k$ square lattice.



octahedron (a) — wheel with 5 spokes (b) — $K_{2,6}$ (c) — 4×4 square lattice (d)

We now show how Euler's formula can be used to prove that certain graphs are *non-planar*. We first derive two corollaries of Theorem 1.2 that give upper bounds for the number of edges of a planar graph.

Corollary 1.1

Let G be a simple connected planar graph with n (≥ 3) vertices and m edges. Then $m \leq 3n - 6$.

Proof

For a plane drawing of a simple connected planar graph G with f faces, it follows from the handshaking lemma for planar graphs that

$$2m \geq 3f,$$

since a simple graph has no loops or multiple edges, so the degree of a face is at least 3. Thus

$$3f \leq 2m.$$

Substituting for f from Euler's formula $f = m - n + 2$, we obtain

$$3m - 3n + 6 \leq 2m,$$

and hence

$$m \leq 3n - 6,$$

as required. ■

Using Corollary 1.1, we can formally prove that the complete graph K_5 is non-planar.

Example 1.1: K_5 is non-planar

The proof is by contradiction.

Suppose that K_5 is planar. Since K_5 is a simple connected graph with 5 vertices and 10 edges, it follows from Corollary 1.1 that

$$10 \leq (3 \times 5) - 6 = 9,$$

which is FALSE. This contradiction shows that K_5 is non-planar. ■

Note that we cannot use Corollary 1.1 to prove that the complete bipartite graph $K_{3,3}$ is non-planar, since $K_{3,3}$ has 6 vertices and 9 edges, and the inequality

$$9 \leq (3 \times 6) - 6 = 12$$

is TRUE. However, we can prove that $K_{3,3}$ is non-planar by using the following corollary for graphs with no triangles.

Corollary 1.2

Let G be a simple connected planar graph with n (≥ 3) vertices, m edges and no triangles. Then $m \leq 2n - 4$.

Proof

For a plane drawing of a simple connected planar graph G with f faces, it follows from the handshaking lemma for planar graphs that

$$2m \geq 4f,$$

since the degree of each face of a simple graph without triangles is at least 4. Thus

$$2f \leq m.$$

Substituting for f from Euler's formula $f = m - n + 2$, we obtain

$$2m - 2n + 4 \leq m,$$

and hence

$$m \leq 2n - 4,$$

as required. ■

Using Corollary 1.2, we can formally prove that the complete bipartite graph $K_{3,3}$ is non-planar.

Example 1.2: $K_{3,3}$ is non-planar

The proof is by contradiction.

Suppose that $K_{3,3}$ is planar. Since $K_{3,3}$ is a simple connected graph with 6 vertices, 9 edges and no triangles, it follows from Corollary 1.2 that

$$9 \leq (2 \times 6) - 4 = 8,$$

which is FALSE. This contradiction shows that $K_{3,3}$ is non-planar. ■

Problem 1.11

Under what conditions do Corollaries 1.1 and 1.2 give equalities

$$m = 3n - 6 \quad \text{and} \quad m = 2n - 4,$$

rather than inequalities?

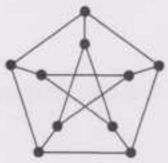
Problem 1.12

(a) Let G be a simple connected planar graph with n (≥ 5) vertices, m edges and shortest cycle length 5. Prove that

$$m \leq \tfrac{5}{3}(n - 2).$$

Hint Use the method of proof of Corollaries 1.1 and 1.2.

(b) Hence show that the Petersen graph is non-planar.



We can similarly prove the following result.

Corollary 1.3

Let G be a simple connected planar graph. Then G contains a vertex of degree 5 or less.

Problem 1.13

Prove Corollary 1.3.

Hint Apply Corollary 1.1 and use a proof by contradiction.

Problem 1.14

Give an example of each of the following:

(a) a simple connected planar graph in which each vertex has degree 5;

(b) a non-simple connected planar graph in which each vertex has degree 6.

1.3 Kuratowski's theorem

The restrictions on the number of edges of a planar graph given in Corollaries 1.1 and 1.2 are useful for showing that certain graphs are *non-planar*. For example, we used them to show that K_5 and $K_{3,3}$ are non-planar. Unfortunately, the method does not work the other way round — there are graphs (such as the Petersen graph) that satisfy these inequalities but are non-planar. Because of this, we now turn our attention to other ways of determining whether a given graph is planar.

The first method we describe involves the insertion of vertices of degree 2 into the edges of a graph G, as shown in the following diagram.

G a subdivision of G

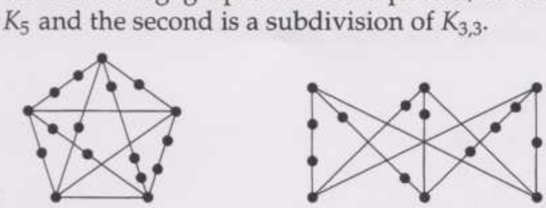
Any graph formed from G in this way is called a **subdivision** of G. Since the insertion of a vertex of degree 2 does not affect the planarity or non-planarity of a graph, we deduce that:

- if G is a planar graph, then every subdivision of G is planar.

This is often stated in the alternative form:

- if G is a subdivision of a non-planar graph, then G is non-planar;

for example, the following graphs are non-planar, since the first is a subdivision of K_5 and the second is a subdivision of $K_{3,3}$.

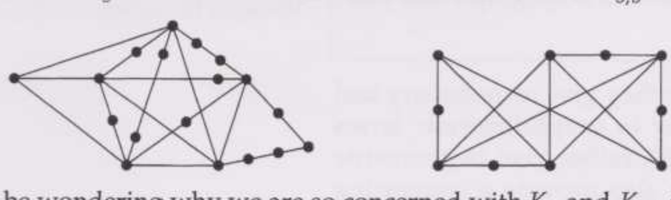


a subdivision of K_5 a subdivision of $K_{3,3}$

It follows from these two observations that

- if a graph G contains a subdivision of K_5 or $K_{3,3}$, then G is non-planar;

for example, the following graphs are non-planar, since the first contains a subdivision of K_5 and the second contains a subdivision of $K_{3,3}$.



You may be wondering why we are so concerned with K_5 and $K_{3,3}$ and their subdivisions. The reason is that *every* non-planar graph is obtained in the way we have just described — namely, by adding vertices and edges to a subdivision of K_5 or $K_{3,3}$. That is,

- if G is a non-planar graph, then G contains a subdivision of K_5 or $K_{3,3}$.

This result appeared in 1930, and is due to the Polish mathematician Kazimierz Kuratowski; the proof is rather long and complicated.

Kazimierz Kuratowski (1896–1980) studied in Glasgow and Warsaw. His main interest was analytic topology, and the paper that earned him a permanent place in graph theory was written from the standpoint of that subject.

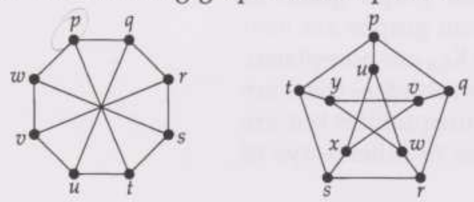
We summarize the above two results formally as follows.

Theorem 1.3: Kuratowski's theorem

A graph is planar if and only if it does not contain a subdivision of K_5 or a subdivision of $K_{3,3}$.

Problem 1.15

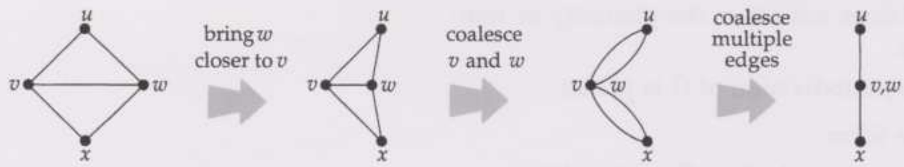
Prove that each of the following graphs is non-planar.



(a) (b)

Hint Use Kuratowski's theorem. For graph (b), consider the subgraph obtained by deleting the two 'horizontal' edges.

Another characterization of planar graphs involves the notion of 'contracting' an edge. This is done by bringing one vertex closer and closer to the other vertex until they coincide, and then coalescing any resulting multiple edges into a single edge. In the following diagrams, we contract the edge vw.



A **contraction** of a graph is the result of a sequence of edge contractions. For example, K_5 is a contraction of the Petersen graph, since it is the result of contracting each of the five 'spokes' (the edges joining the inner and outer 5-cycles).

We now state the following analogue of Kuratowski's theorem.

Theorem 1.4

A graph is planar if and only if it does not contain a subgraph that has K_5 or $K_{3,3}$ as a contraction.

We omit the proof.

The importance of Theorems 1.3 and 1.4 is that they give us necessary and sufficient conditions for a graph to be planar in graph-theoretic terms (subgraph, subdivision, contraction of a graph), rather than in geometric terms (crossing, drawing in the plane). They also provide a convincing demonstration that a given graph is non-planar, if we happen to spot a subgraph that is a subdivision of K_5 or $K_{3,3}$, or a subgraph that has K_5 or $K_{3,3}$ as a contraction.

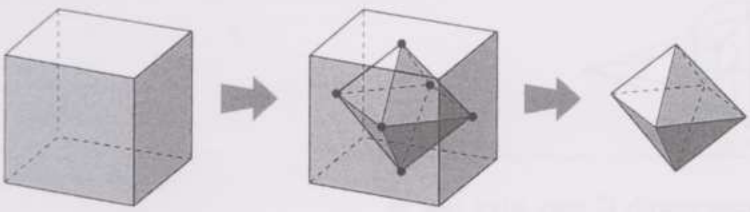
However, Theorems 1.3 and 1.4 do not provide an easy way of showing that a given graph is planar, since this would involve looking at a large

number of subgraphs and verifying that none of them is a subdivision of K_5 or $K_{3,3}$, or contains K_5 or $K_{3,3}$ as a contraction. For this reason, no currently used algorithm for testing the planarity of a graph is based on either of these theorems.

We describe a method for testing the planarity of a graph in Section 1.5.

1.4 Duality

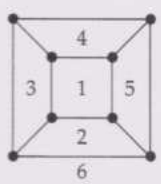
We next introduce the idea of *duality* for plane drawings of planar graphs. Recall from *Design 1* that we form the dual of a polyhedron by placing a new vertex at the centre of each face and joining the pairs of new vertices in adjacent faces. For example, the dual of a cube is an octahedron, as shown below.



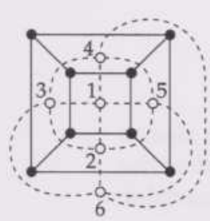
cube
8 vertices
6 faces

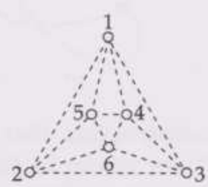
octahedron
8 faces
6 vertices

We can carry out a similar procedure on the graphs of the polyhedra. For example, let us take the graph of the cube. If we place a new vertex within each face (including the infinite face) and join the pairs of new vertices in adjacent faces, we obtain a new graph which is the graph of the octahedron, and *vice versa*, as follows.



cube
8 vertices
6 faces





octahedron
8 faces
6 vertices

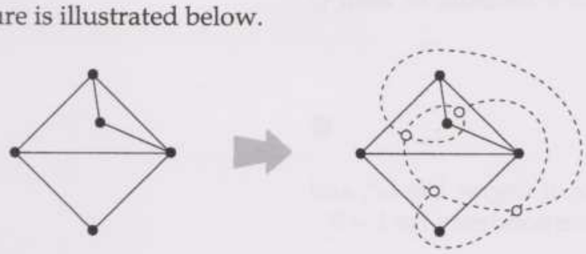
The new vertices are represented by small circles, and the edges joining them are indicated by dashed lines.

More generally, for any connected planar graph G, we define a dual graph G^* as follows.

> **Definition**
>
> Let G be a connected planar graph. Then a **dual graph** G^* is constructed from a plane drawing of G, as follows.
>
> - Draw one new vertex in each face of the plane drawing — these are the vertices of G^*.
> - For each edge e of the plane drawing, draw a line joining the vertices of G^* in the faces on either side of e — these lines are the edges of G^*.

We always assume that we have been presented with a plane drawing of G.

The procedure is illustrated below.

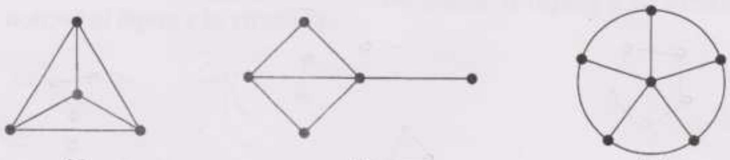


G G^*

As before, the vertices of G^* are represented by small circles, and the edges joining them are indicated by dashed lines.

Problem 1.16

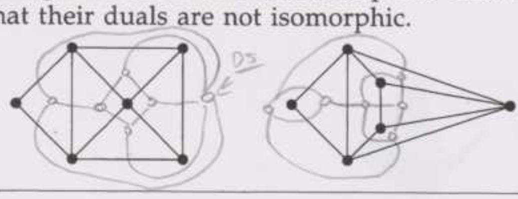
Draw the dual of each of the following plane drawings of planar graphs.



(a) (b) (c)

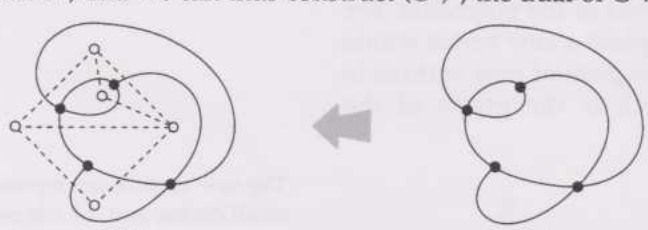
Problem 1.17

The following diagrams show two different plane drawings of a planar graph. Show that their duals are not isomorphic.



Note that different plane drawings of a planar graph G may give rise to non-isomorphic dual graphs G^*, as we saw in the above problem.

Note also that, if G is a plane drawing of a connected planar graph, then so is its dual G^*, and we can thus construct $(G^*)^*$, the dual of G^*.



$(G^*)^*$ G^*

The above diagrams demonstrate that the construction that gives rise to G^* from G can be reversed to give G from G^*; for example, the dual of the octahedron graph is the cube graph. It follows that $(G^*)^*$ is isomorphic to G.

There is a simple relationship between the numbers of vertices, edges and faces of a plane drawing of a graph and its dual. In the above example, G has 5 vertices, 7 edges and 4 faces (including the infinite face), and G^* has 4 vertices, 7 edges and 5 faces.

In general, we have the following result.

Theorem 1.5

Let G be a plane drawing of a connected planar graph with n vertices, m edges and f faces. Then G^* has f vertices, m edges and n faces.

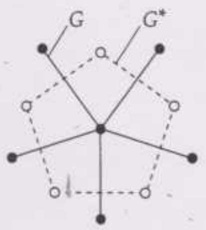
Proof

It follows directly from the construction of G^* that G^* has f vertices and m edges. If G^* has f^* faces, then, by applying Euler's formula to both G and G^*, we obtain

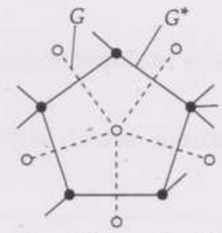
for G: $n - m + f = 2$; for G^*: $f - m + f^* = 2$.

Comparing these, we obtain $f^* = n$, as required. ■

In fact, a vertex of degree k in G corresponds to a face of degree k in G^*, and *vice versa*. The following diagram illustrates this correspondence for $k = 5$.



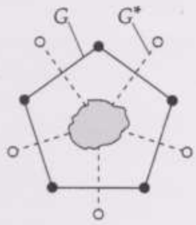
vertex of degree 5 in G corresponds to face of degree 5 in G^*



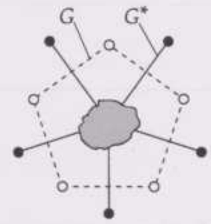
face of degree 5 in G^* corresponds to vertex of degree 5 in G

Further, a cycle of length k in G corresponds to a cutset with k edges in G^*, and *vice versa*. Again, we illustrate this correspondence for $k = 5$.

A *cutset* is a set of edges whose removal disconnects a graph, and does not include 'redundant' edges.



cycle of length 5 in G corresponds to cutset of G^* with 5 edges



cutset of G^* with 5 edges corresponds to cycle of length 5 in G

To obtain the first of the above correspondences, we take a cycle in G (with solid edges); the corresponding edges of G^* (the dashed edges) form a cutset whose removal separates the set of vertices inside the cycle from those outside. To obtain the second correspondence, we interchange the roles of G and G^*.

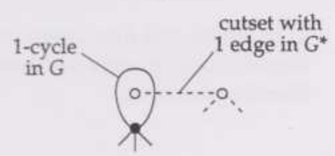
We summarize these correspondences as follows.

plane drawing G		dual graph G^*
an edge of G	corresponds to	an edge of G^*
a vertex of degree k in G	corresponds to	a face of degree k in G^*
a face of degree k in G	corresponds to	a vertex of degree k in G^*
a cycle of length k in G	corresponds to	a cutset of G^* with k edges
a cutset of G with k edges	corresponds to	a cycle of length k in G^*

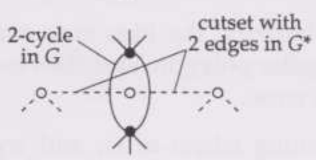
We can use these correspondences to obtain new results from old ones. For example, we can reword Corollary 1.1 as follows.

> Let G be a connected planar graph with n (≥ 3) vertices and m edges, and with no loops or multiple edges. Then $m \leq 3n - 6$.

Now, loops (cycles of length 1) and pairs of multiple edges (cycles of length 2) in G correspond to cutsets with 1 and 2 edges in G^*.



loop in G



2 multiple edges in G

The above correspondence therefore gives the following theorem.

> **Theorem 1.6**
>
> Let G^* be a connected planar graph with f faces and m edges, and with no cutsets with 1 or 2 edges. Then $m \leq 3f - 6$.

Conversely, we can *dualize* Theorem 1.6 to obtain Corollary 1.1.

Similarly, we can reword Corollary 1.3 as follows.

> Let G be a connected planar graph with no loops or multiple edges. Then G has a vertex of degree 5 or less.

Dualizing this result, we deduce the following theorem, which will be needed in Section 2.

Theorem 1.7

Let G^* be a connected planar graph with no cutsets with 1 or 2 edges. Then G^* has a face of degree 5 or less.

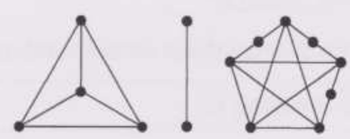
Problem 1.18

Dualize Corollary 1.2.

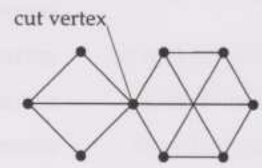
1.5 Testing for planarity

In many practical applications it is important to test whether a given graph is planar. There are several methods for this purpose in current use, and we present one of these, the *cycle method*, here. It can be applied to any small graph containing a Hamiltonian cycle. There exist fast algorithms which work for all graphs, but they are too complicated to be included here.

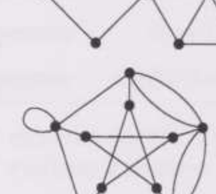
Before presenting the cycle method, we make a few observations that simplify the task of determining whether a given graph is planar.



- If the graph is disconnected, then it is planar if and only if each of its components is planar; for example, the graph in the margin is non-planar, because one of its components is a subdivision of K_5.



- If the graph has a cut vertex (a vertex whose removal disconnects the graph), then it is planar if and only if each of the subgraphs obtained when the graph is 'broken apart' at the cut vertex is planar; for example, the graph in the margin is non-planar, because one of these subgraphs is $K_{3,3}$.



- If the graph has loops or multiple edges, then it is planar if and only if the graph obtained by removing the loops and coalescing the multiple edges is planar; for example, the graph in the margin is non-planar, because the resulting graph is the Petersen graph.

Using these observations, we can sometimes reduce a given graph to a number of smaller graphs that we can deal with more easily.

Cycle method for planarity testing

In the following discussion we consider only graphs that have a Hamiltonian cycle.

Given a graph G that we wish to test for planarity, the idea is to find a Hamiltonian cycle, to draw this cycle as a regular polygon, and then to try to draw the remaining edges so that no edges cross.

Having chosen a cycle C, we list the remaining edges of G, and try to divide them into two disjoint sets A and B, as follows:

Recall that two subsets are *disjoint* if they have no element in common.

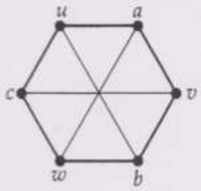
> A is a set of edges which can be drawn *inside* C without crossing;
>
> B is a set of edges which can be drawn *outside* C without crossing.

If this is possible, the graph G is planar, and we can use the sets A and B to obtain a plane drawing of G. If this is not possible, the graph G is non-planar.

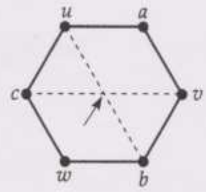
See Section 1.1.

You met an example of this procedure earlier when we tested the complete bipartite graph $K_{3,3}$ for planarity. We started by noting that $K_{3,3}$ has a Hamiltonian cycle C of length 6, which we drew in the plane as a regular

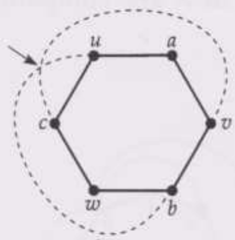
hexagon *uavbwcu*. We then tried to draw in the three remaining edges *ub*, *vc* and *wa*; but only one of these edges can lie inside C, and only one can lie outside C, otherwise two of them cross. Thus, if we put *ub* in the set A and *vc* in the set B, then we cannot allocate *wa* to either set; it follows that we cannot draw in all three edges without crossings, and so the graph $K_{3,3}$ is non-planar.



$K_{3,3}$
C is cycle *uavbwcu*



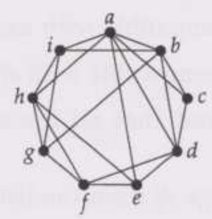
ub, *vc* cross
inside C



ub, *vc* cross
outside C

We say that the edges *ub* and *vc* are *incompatible*, since they cannot both be drawn inside C, or both be drawn outside C, without crossing. Similarly, the edges *ub* and *wa* are incompatible, as are the edges *vc* and *wa*.

The following example shows how this idea of incompatible edges can be used to test the planarity of more complicated graphs.

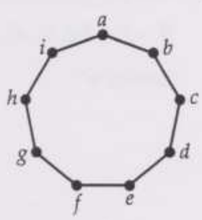


graph G

Example 1.3

Determine whether the graph G shown in the margin is planar.

The first step is to choose a suitable cycle C in G. In this example, it is natural to choose the Hamiltonian cycle *abcdefghia*.



cycle C

ac	*bd*	*df*
ad	*bg*	*eh*
ae	*bi*	*fh*
ah		*gi*

edges which do not belong to C

We list the edges which do not belong to C. list: *ac, ad, ae, ah, bd, bg, bi, df, eh, fh, gi*

We put the first edge in the list, *ac*, in a set A. $A = \{ac, \ldots\}$

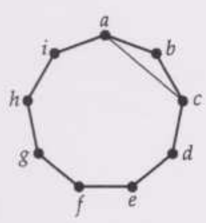
We delete this edge from the list. list: *ad, ae, ah, bd, bg, bi, df, eh, fh, gi*

The edge *ac* is incompatible with *bd*, *bg* and *bi*, so we put the edges *bd*, *bg* and *bi* in a set B. $B = \{bd, bg, bi, \ldots\}$

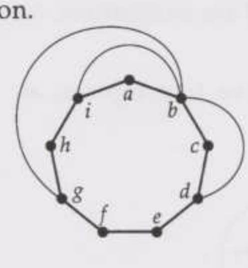
We check and find that all the edges in B are *compatible with each other* — that is, they can all be drawn outside C without crossing. CHECK ✔

We delete the edges *bd*, *bg* and *bi* from the list. list: *ad, ae, ah, df, eh, fh, gi*

We now have the following situation.



edges in A:
drawn inside C



edges in B:
drawn outside C

We now take each edge of B in turn.

The edge *bd* is compatible with each edge in the list.

The edge *bg* is incompatible with *ad*, *ae*, *eh* and *fh*, so we put the edges *ad*, *ae*, *eh* and *fh* into A. $A = \{ac, ad, ae, eh, fh, \ldots\}$

We check and find that all the edges in A are compatible with each other. CHECK ✔

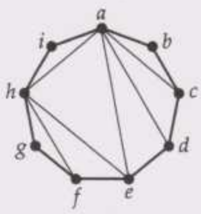
We delete the edges *ad*, *ae*, *eh* and *fh* from the list. list: *ah*, *df*, *gi*

The edge *bi* is incompatible with *ah*, so we put the edge *ah* into A. $A = \{ac, ad, ae, eh, fh, ah, \dots\}$

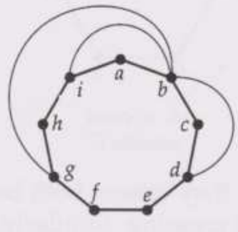
We check and find that all the edges in A are compatible with each other. CHECK ✔

We delete the edge *ah* from the list. list: *df*, *gi*

The situation is now as follows.



edges in A



edges in B

We now take each new edge of A in turn.

The edge *ad* is compatible with each edge in the list.

The edge *ae* is incompatible with *df*, so we put the edge *df* into B. $B = \{bd, bg, bi, df, \dots\}$

We check and find that all the edges in B are compatible with each other. CHECK ✔

We delete the edge *df* from the list. list: *gi*

The edge *eh* in A is incompatible with *gi*, so we put the edge *gi* into B. $B = \{bd, bg, bi, df, gi\}$

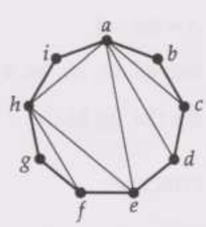
We check and find that all the edges in B are compatible with each other. CHECK ✔

We delete the edge *gi* from the list.

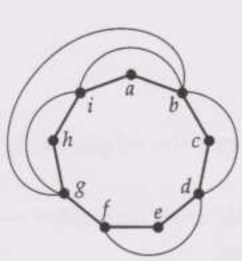
The list is now empty, and we have:

$A = \{ac, ad, ae, eh, fh, ah\}$;

$B = \{bd, bg, bi, df, gi\}$.



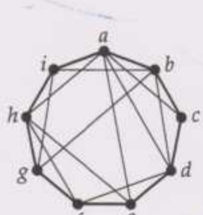
edges in A



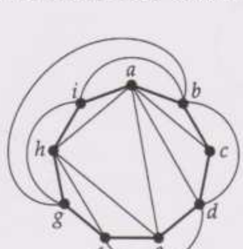
edges in B

All the edges in A are compatible and all the edges in B are compatible, so G is planar.

To obtain a plane drawing of G, we combine the above two figures, as follows.



graph G

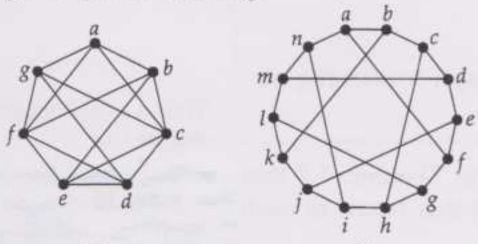


plane drawing of G

■

Problem 1.19

Use the cycle method to determine whether each of the following graphs is planar. If it is, give a plane drawing.



(a) (b)

1.6 Computer activities

The computer activities for this section are described in the *Computer Activities Booklet*.

After studying this section, you should be able to:

- explain the terms *planar graph, non-planar graph, plane drawing, face, infinite face, degree of a face, subdivision of a graph* and *contraction of a graph*;
- state and use the handshaking lemma for planar graphs;
- state and use Euler's formula and the corollaries to Theorem 1.2;
- understand the statement of Kuratowski's theorem;
- explain the term *dual graph* and describe its properties;
- use the cycle method to determine whether a given graph is planar.

2 Colouring maps

In the television programme we outline a proof of the four colour theorem for maps.

Theorem 2.1: four colour theorem for maps

The countries (faces) of any map can be coloured with four (or fewer) colours in such a way that neighbouring countries are coloured differently.

2.1 Four colour problem

This subsection is a television subsection. The printed material associated with the programme is given in the *Television Notes Booklet*. You are advised to look at this material before watching the programme.

2.2 Two proofs

In this subsection, we prove the six colour theorem and five colour theorem for maps by the method of mathematical induction.

The method of proof by mathematical induction is explained in *Graphs 1*.

Before proceeding, we recall the formal definition of a *map* given in the television programme.

> **Definition**
>
> A **map** is a plane drawing of a connected planar graph containing no cutsets with 1 or 2 edges.

Thus a map contains no vertices of degree 1 or 2.

We refer to a face of a map as a *country*.* It follows from Theorem 1.7 that every map has a country of degree 5 or less. We use this result in each proof.

* This definition of a map implies that the outside region (the infinite face) is a country, although it's often not regarded as such. As explained on page 30 of television notes, it is immaterial whether or not this region is included in a colouring of the map.

We now prove the six colour theorem for maps.

> **Theorem 2.2: six colour theorem for maps**
>
> The countries of any map can be coloured with six (or fewer) colours in such a way that neighbouring countries are coloured differently.

Proof

We prove this result by mathematical induction on n, the number of countries of the map M.

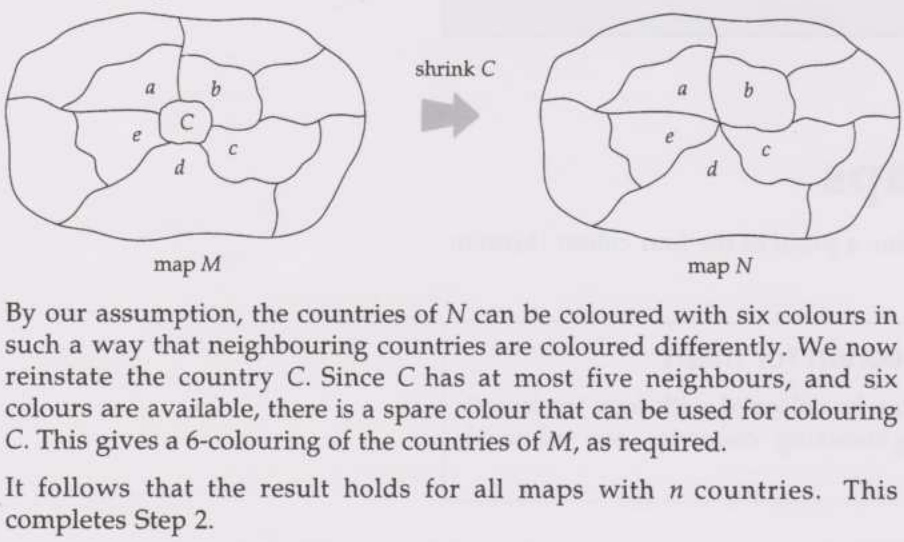
STEP 1 *Show that the result holds for the map with one country.*

This is trivially true.

In fact, the result clearly holds for all maps with up to six countries, since a different colour can be used for each country.

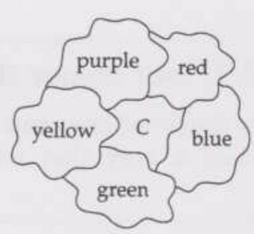
STEP 2 *Show that, for each positive integer n, if the result holds for maps with fewer than n countries, then it must also hold for maps with n countries.*

We assume that the result holds for all maps with fewer than n countries, and let M be a map with n countries. It follows from Theorem 1.7 that M contains a country C of degree 5 or less. If we shrink C to a point, then the resulting map N has fewer than n countries.



map M map N

By our assumption, the countries of N can be coloured with six colours in such a way that neighbouring countries are coloured differently. We now reinstate the country C. Since C has at most five neighbours, and six colours are available, there is a spare colour that can be used for colouring C. This gives a 6-colouring of the countries of M, as required.



It follows that the result holds for all maps with n countries. This completes Step 2.

Therefore, by the principle of mathematical induction, the result holds for all maps with n countries, for each positive integer n. It therefore holds for all maps. ■

With a little more effort, we can prove the following stronger theorem.

> **Theorem 2.3: five colour theorem for maps**
>
> The countries of any map can be coloured with five (or fewer) colours in such a way that neighbouring countries are coloured differently.

Proof

We prove this result by mathematical induction on n, the number of countries of the map M.

STEP 1 *Show that the result holds for the map with one country.*

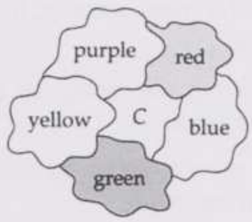
This is trivially true.

In fact, the result clearly holds for all maps with up to five countries, since a different colour can be used for each country.

STEP 2 *Show that, for each positive integer n, if the result holds for maps with fewer than n countries, then it must also hold for maps with n countries.*

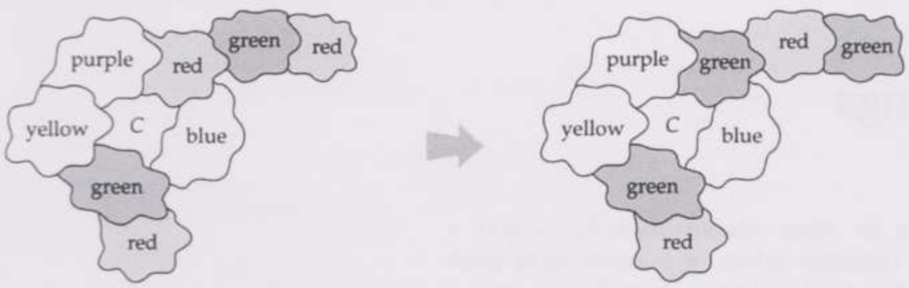
We assume that the result holds for all maps with fewer than n countries, and let M be a map with n countries. It follows from Theorem 1.7 that M contains a country C of degree 5 or less. If we shrink C to a point, then the resulting map N has fewer than n countries.

By our assumption, the countries of N can be coloured with five colours in such a way that neighbouring countries are coloured differently. We now reinstate the country C. Since C has at most five neighbours, and five colours are available, there is a spare colour that can be used for colouring C, *unless C is surrounded by five countries of different colours*; in this case, there is no spare colour that can be used to colour C.



In order to overcome this difficulty, we consider, for example, just the red and green countries adjacent to C, and investigate whether there is a path of red and green countries between the adjacent red one and the adjacent green one. The two situations that can arise are shown below.

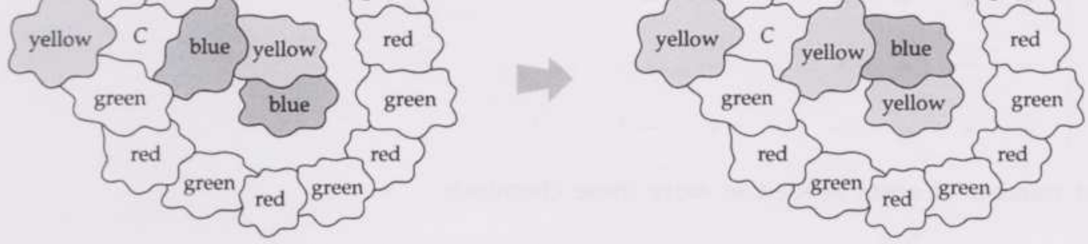
In case (a), all the red and green countries reachable from the adjacent red one are different from those reachable from the adjacent green one, so there is no such red–green path. In this case, we interchange the colours in the red–green part at the top, say, as shown below.



case (a): interchange red and green on top right; colour C red

This replaces the red country adjacent to C by a green one, so that C can now be coloured red. This completes the 5-colouring of the map M in this case.

In case (b), the two red–green parts link up, so there is a red–green path, and interchanging the colours does not help us, as the country C is still adjacent to a red country and a green one. In this case, there can be no path of blue and yellow countries between the blue and yellow countries adjacent to C, because the red–green path 'gets in the way'. We can therefore interchange the colours in the blue–yellow part on the right-hand side, say, as shown below.



case (b): interchange blue and yellow on right; colour C blue

This replaces the blue country adjacent to C by a yellow one, so that C can now be coloured blue. This completes the 5-colouring of the map M in this case.

It follows that the result holds for all maps with n countries. This completes Step 2.

Therefore, by the principle of mathematical induction, the result holds for all maps with n countries, for each positive integer n. It therefore holds for all maps. ■

After studying this section, you should be able to:

- explain what is meant by a *map*;
- outline the main steps in the proof of the four colour theorem for maps;
- explain what are meant by an *unavoidable set* and a *reducible configuration*;
- outline proofs of the six colour theorem and the five colour theorem for maps.

3 Vertex colourings and decompositions

In this section, we consider problems involving the colouring of the *vertices* of a graph.

3.1 Vertex colourings

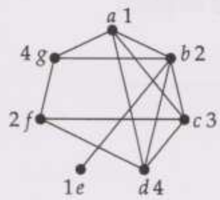
Example 3.1: storing chemicals

A chemical manufacturer wishes to store chemicals $a, b, \ldots, g$ in a warehouse. Some chemicals react violently when in contact with each other, and the manufacturer decides to divide the warehouse into a number of areas so as to separate dangerous pairs of chemicals. In the following table, an asterisk indicates those pairs of chemicals that must be kept separate.

	a	b	c	d	e	f	g
a	–	*	*	*	–	–	*
b	*	–	*	*	*	–	*
c	*	*	–	*	–	*	–
d	*	*	*	–	–	*	–
e	–	*	–	–	–	–	–
f	–	–	*	*	–	–	*
g	*	*	–	–	–	*	–

What is the smallest number of areas needed to store these chemicals safely?

We note first that chemicals a, b, c and d must all be in separate areas, and so at least four areas are necessary. We can see that four areas are sufficient from the following graph. The vertices correspond to the seven chemicals, and two vertices are joined by an edge whenever the corresponding chemicals must be kept separate; if we colour the vertices with the minimum number of colours so that adjacent vertices are coloured differently, we find that 4 colours are needed. The four colours 1, 2, 3, 4 correspond to the four areas.



Thus we can split the set of chemicals into four disjoint subsets corresponding to the four areas:

$\{a, e\}, \{b, f\}, \{c\}, \{d, g\}.$ ■

Other solutions are possible.

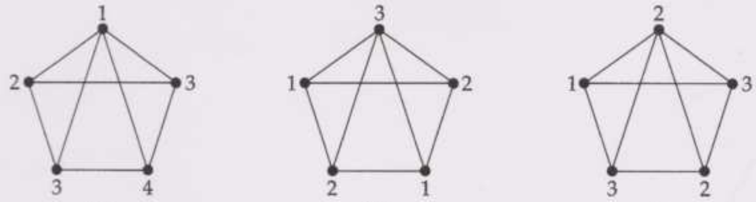
The assignment of colours to chemicals in the above discussion illustrates the following definitions.

> ### Definitions
>
> Let G be a simple graph. A **k-colouring** of G is an assignment of k colours to the vertices of G in such a way that adjacent vertices are assigned different colours. If G has a k-colouring, then G is **k-colourable**.
>
> The **chromatic number** of G, denoted by $\chi(G)$, is the smallest number k for which G is k-colourable.

In the above chemical storage problem, the colours correspond to the areas. Thus, the above graph has chromatic number $\chi(G) = 4$.

Note that the above definitions are given only for *simple* graphs. Loops must be excluded since, in any k-colouring, the vertices at the ends of each edge must be assigned different colours, and so the vertex at both ends of a loop would have to be assigned a different colour from itself. We also exclude multiple edges, since the presence of one edge between two vertices forces them to be coloured differently, and the addition of further edges between them is then irrelevant to the colouring. *We therefore restrict our attention to simple graphs.*

We usually show a k-colouring by writing the numbers 1, 2, ..., k next to the appropriate vertices. For example, diagrams (a) and (b) below illustrate a 4-colouring and a 3-colouring of a graph G with five vertices; note that diagram (c) is *not* a 3-colouring of G, since the two vertices coloured 2 are adjacent.



(a) (b) (c)

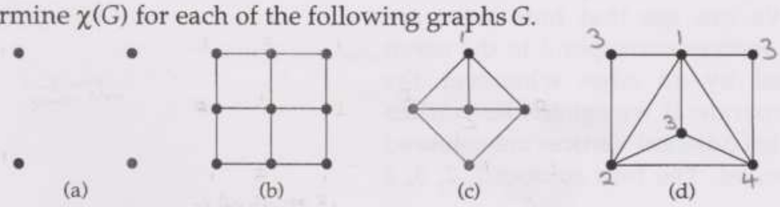
Since we have shown that G has a 3-colouring, it follows that $\chi(G) \leq 3$. Also, G contains three mutually adjacent vertices (forming a triangle) which must be assigned different colours, so $\chi(G) \geq 3$. Combining these inequalities, we obtain $\chi(G) = 3$.

Thus 3 is an *upper bound* for $\chi(G)$.

Thus 3 is a *lower bound* for $\chi(G)$.

Problem 3.1

Determine $\chi(G)$ for each of the following graphs G.



Hint For each graph, you need to devise a suitable colouring *and* explain why there is no colouring with fewer colours.

Problem 3.2

What can you say about the graphs G for which

(a) $\chi(G) = 1$?   (b) $\chi(G) = 2$?

Problem 3.3

Write down the chromatic number of each of the following graphs:

(a) the complete graph K_n;

(b) the complete bipartite graph $K_{r,s}$;

(c) the cycle graph C_n ($n \geq 3$);

(d) a tree.

Problem 3.4

Decide whether each of the following statements about a graph G is TRUE or FALSE, and give a proof or counter-example as appropriate.

(a) If G contains the complete graph K_r as a subgraph, then $\chi(G) \geq r$.

(b) If $\chi(G) \geq r$, then G contains the complete graph K_r as a subgraph.

Given a particular graph G, how can we determine its chromatic number? We have seen that an upper bound for $\chi(G)$ may be obtained by construction:

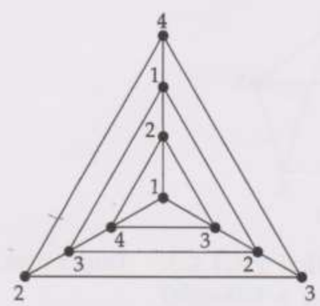
> to obtain an upper bound for $\chi(G)$, construct an explicit colouring for the vertices of G.

A lower bound for $\chi(G)$ may be obtained using the result of Problem 3.4(a):

> to obtain a lower bound for $\chi(G)$, find the number of vertices in the largest complete subgraph in G.

For example, if G contains K_3 (a triangle), then $\chi(G) \geq 3$.

If we can find an upper bound and a lower bound which are the same, then $\chi(G)$ is equal to this common value. For example, the vertices of the graph G below can be coloured with four colours, as shown, and so $\chi(G) \leq 4$. But G cannot be coloured with fewer than four colours, since G contains the complete graph K_4, and so $\chi(G) \geq 4$. Combining these two inequalities, we deduce that $\chi(G) = 4$. A 4-colouring is shown below.



Note that if a graph G has n vertices, then $\chi(G) \leq n$. However, this upper bound is usually rather poor, except when G has many edges.

This inequality becomes an equality ($\chi(G) = n$) when G is the complete graph K_n.

We can improve on this upper bound considerably if we know the largest vertex degree in G, as our next theorem shows. First, try the following problem.

Problem 3.5

Draw two non-isomorphic simple connected graphs G with five vertices and maximum vertex degree d for which $\chi(G) = d + 1$.

Theorem 3.1

Let G be a simple graph whose maximum vertex degree is d. Then

$\chi(G) \leq d + 1.$

Proof

We prove this result by mathematical induction on n, the number of vertices of G.

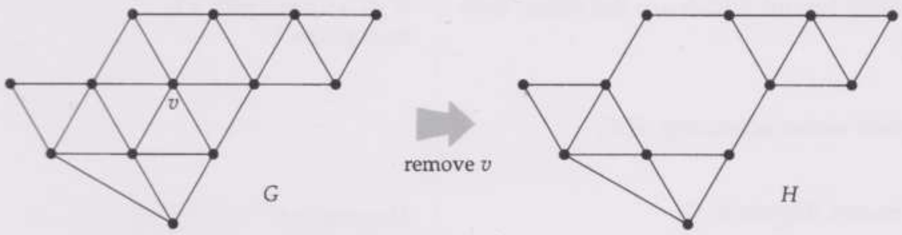
An alternative method of proof, using a greedy algorithm, is discussed in Section 3.2.

STEP 1 *Show that the result holds for the simple graph with one vertex.*

We have $\chi(K_1) = 1$ and $d = 0$, and so the result is true in this case.

STEP 2 *Show that, for each positive integer n, if the result holds for all simple graphs with fewer than n vertices, then it must also hold for all simple graphs with n vertices.*

We assume that the result holds for all simple graphs with fewer than n vertices. Let G be a simple graph with n vertices and maximum vertex degree d, and let H be any graph obtained from G by removing a vertex v and the edges incident with it.



Since H has fewer than n vertices and maximum vertex degree d (or less), it follows from our assumption that $\chi(H) \leq d + 1$ — that is, the graph H is $(d + 1)$-colourable. We can now obtain a $(d + 1)$-colouring of G by colouring v with any colour not assigned to the (at most d) vertices adjacent to v.

These vertices can be coloured with at most d colours.

It follows that $\chi(G) \leq d + 1$, and so the result holds for all simple graphs with n vertices. This completes Step 2.

Therefore, by the principle of mathematical induction, the result holds for all simple graphs with n vertices, for each positive integer n. It therefore holds for all simple graphs. ■

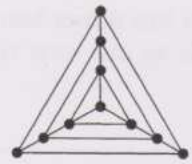
With a lot more effort, we can prove the following slightly stronger theorem; we omit the proof.

Theorem 3.2: Brooks' theorem

Let G be a connected simple graph whose maximum vertex degree is d. If G is neither a cycle graph with an odd number of vertices, nor a complete graph, then $\chi(G) \leq d$.

L. Brooks proved this theorem in 1941. Note that the two exceptions, the cycle graph with an odd number of vertices and the complete graph, are the examples we gave in the solution to Problem 3.5.

To illustrate the use of Brooks' theorem, we consider again the graph G in the margin. We have already observed that $\chi(G) \geq 4$, since G contains the complete graph K_4. On the other hand, G satisfies the conditions of Brooks' theorem, with $d = 4$, and so $\chi(G) \leq 4$. It follows that $\chi(G) = 4$.



Unfortunately, the situation is not always as satisfactory as this. In particular, if G contains a few vertices of high degree, then the bound given by Brooks' theorem may be very poor. For example, if G is the bipartite graph $K_{1,12}$, then Brooks' theorem gives the upper bound $\chi(G) \leq 12$, whereas the actual value of $\chi(G)$ is 2.

$K_{1,12}$

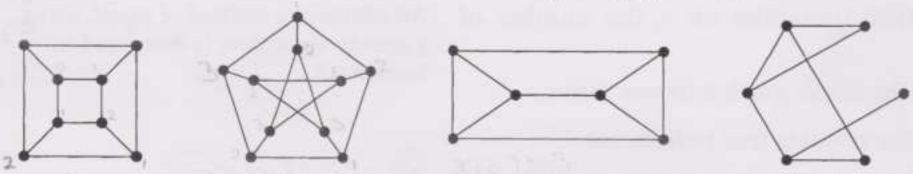
Problem 3.6

For each of the following graphs G, write down:

the lower bound for $\chi(G)$ given by the size of the largest complete subgraph in G;

the upper bound for $\chi(G)$ given by Brooks' theorem;

the actual value of $\chi(G)$, and a colouring using $\chi(G)$ colours.



(a) (b) (c) (d)

We summarize the above results as follows.

To find the chromatic number $\chi(G)$ of a simple graph G

Try to find an upper bound and a lower bound which are the same; then $\chi(G)$ is equal to this common value.

If $\chi(G) \leq k$ and $\chi(G) \geq k$, then $\chi(G) = k$.

possible upper bounds for $\chi(G)$

- the number of colours in an explicit vertex colouring of G;
- the number n of vertices in G;
- $d + 1$, where d is the maximum vertex degree in G; Theorem 3.1
- d, where d is the maximum vertex degree in G, provided that G is not C_n (for odd n) or K_n. Brooks' theorem

possible lower bound for $\chi(G)$

- the number of vertices in the largest complete subgraph in G.

Colouring planar graphs

It seems natural to conjecture that the more complicated a graph, the more colours are needed to colour its vertices. In this subsection we show that, if the graph is planar, then this conjecture is false — the chromatic number of a planar graph is 'small'.

Our first result of this type shows that every planar graph is 6-colourable.

Theorem 3.3: six colour theorem for planar graphs

The vertices of any simple connected planar graph G can be coloured with six (or fewer) colours in such a way that adjacent vertices are coloured differently.

Although this theorem can be obtained by dualizing Theorem 2.2, we consider it useful to give a direct proof. You should compare this proof with that of Theorem 2.2.

Proof

We prove this result by mathematical induction on n, the number of vertices of G.

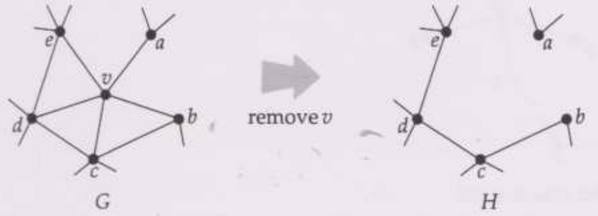
STEP 1 *Show that the result holds for the graph with one vertex.*

This is trivially true.

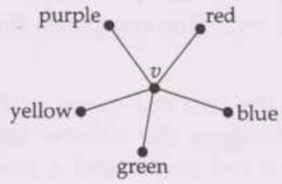
In fact, the result clearly holds for all graphs with up to six vertices, since a different colour can be used for each vertex.

STEP 2 *Show that, for each positive integer n, if the result holds for graphs with fewer than n vertices, then it must also hold for graphs with n vertices.*

We assume that the result holds for all simple connected planar graphs with fewer than n vertices, and let G be a simple connected planar graph with n vertices. It follows from Corollary 1.3 that G contains a vertex v of degree 5 or less. If we remove v and its incident edges, then the resulting planar graph H has fewer than n vertices.



By our assumption, the vertices of H (or of each component of H, if H is disconnected) can be coloured with six colours in such a way that adjacent vertices are coloured differently. We now reinstate the vertex v. Since v has at most five neighbours, and six colours are available, there is a spare colour that can be used for colouring v. This gives a 6-colouring of the vertices of G, as required.



It follows that the result holds for all simple connected planar graphs with n vertices. This completes Step 2.

Therefore, by the principle of mathematical induction, the result holds for all simple connected planar graphs with n vertices, for each positive integer n. It therefore holds for all simple connected planar graphs. ■

With a little more effort, we can prove the following stronger theorem.

Theorem 3.4: five colour theorem for planar graphs

The vertices of any simple connected planar graph G can be coloured with five (or fewer) colours in such a way that adjacent vertices are coloured differently.

Proof

You should compare this proof with that of Theorem 2.3.

We prove this result by mathematical induction on n, the number of vertices of G.

STEP 1 *Show that the result holds for the graph with one vertex.*

This is trivially true.

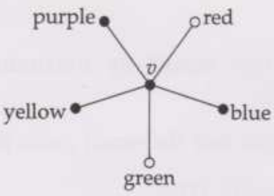
In fact, the result clearly holds for all graphs with up to five vertices, since a different colour can be used for each vertex.

STEP 2 *Show that, for each positive integer n, if the result holds for graphs with fewer than n vertices, then it must also hold for graphs with n vertices.*

We assume that the result holds for all simple connected planar graphs with fewer than n vertices, and let G be a simple connected planar graph with n vertices. It follows from Corollary 1.3 that G contains a vertex v of degree 5 or less. If we remove v and its incident edges, then the resulting planar graph H has fewer than n vertices.

By our assumption, the vertices of H (or of each component of H, if H is disconnected) can be coloured with five colours in such a way that adjacent vertices are coloured differently. We now reinstate the vertex v. Since there are at most five vertices adjacent to v, and five colours are

available, there is a spare colour that can be used for colouring v, *unless v is surrounded by five vertices of different colours*; in this case, there is no spare colour that can be used to colour v.



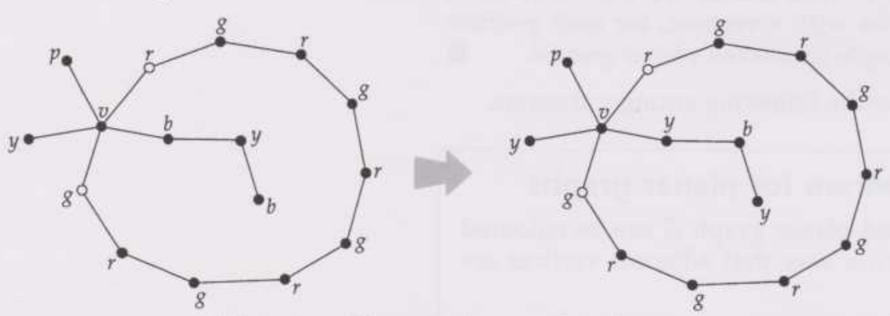
In order to overcome this difficulty, we consider, for example, just the red and green vertices adjacent to v, and investigate whether there is a path of red and green vertices between the adjacent red vertex and the adjacent green vertex. There are two situations that can arise.

In case (a), all the red and green vertices reachable from the adjacent red vertex are different from those reachable from the adjacent green vertex, so there is no such red–green path. In this case, we interchange the colours in the red–green part at the top, say, as shown below.

case (a): interchange red and green on top right; colour v red

This replaces the red vertex adjacent to v by a green one, so that v can now be coloured red. This completes the 5-colouring of the vertices of G in this case.

In case (b), the two red–green paths link up, so there is a red–green path, and interchanging the colours does not help us, as the vertex v is still adjacent to a red vertex and a green vertex. In this case, there can be no path of blue and yellow vertices between the blue and yellow vertices adjacent to v, because the red–green path 'gets in the way'. We can therefore interchange the colours in the blue–yellow part on the right-hand side, say, as shown below.



case (b): interchange blue and yellow on right; colour v blue

This replaces the blue vertex adjacent to v by a yellow vertex, so that v can now be coloured blue. This completes the 5-colouring of the vertices of G in this case.

It follows that the result holds for all simple connected planar graphs with n vertices. This completes Step 2.

Therefore, by the principle of mathematical induction, the result holds for all simple connected planar graphs with n vertices, for each positive integer n. It therefore holds for all simple connected planar graphs. ■

We now state without proof the four colour theorem for planar graphs.

Theorem 3.5: four colour theorem for planar graphs

The vertices of any simple connected planar graph can be coloured with four (or fewer) colours in such a way that adjacent vertices are coloured differently.

The history of the dual version of this theorem, and an outline of its proof, were discussed in Section 2.1.

3.2 Algorithm for vertex colouring

It is natural to ask whether there are efficient algorithms for colouring the vertices of a graph. Unfortunately, no such efficient algorithms are known. We must therefore seek either inefficient algorithms that give the correct value for the number of colours needed, or heuristic algorithms that are efficient but give only an approximation to the correct value. In this subsection we present a heuristic algorithm — a straightforward colouring algorithm that usually gives good answers.

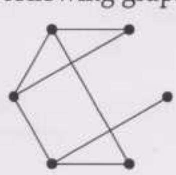
The method we describe is a greedy algorithm, and can be stated as follows.

> **Greedy algorithm for vertex colouring**
>
> START with a graph G and list of colours $1, 2, 3, \ldots$.
>
> STEP 1 Label the vertices $a, b, c, \ldots$ in any manner.
>
> STEP 2 Identify the uncoloured vertex labelled with the earliest letter in the alphabet.
>
> Colour it with the first colour in the list not used for any adjacent coloured vertex.
>
> Repeat Step 2 until all the vertices are coloured, then STOP.
>
> A vertex colouring of G has been obtained. The number of colours used depends on the labelling chosen for the vertices in Step 1.

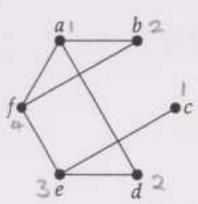
We illustrate the use of the algorithm by two examples using the same graph with different labellings.

Example 3.1A

Find a vertex colouring of the following graph G.



STEP 1 We label the vertices $a, \ldots, f$ as follows.



STEP 2 We colour vertex a with colour 1.

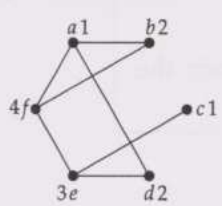
STEP 2 We colour vertex b with colour 2.

STEP 2 We colour vertex c with colour 1.

STEP 2 We colour vertex d with colour 2.

STEP 2 We colour vertex e with colour 3.

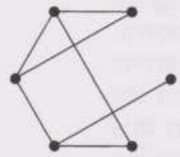
STEP 2 We colour vertex f with colour 4.



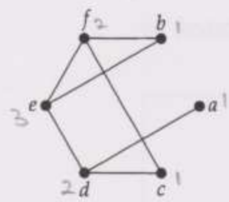
All the vertices are now coloured, so we STOP. We thus obtain the 4-colouring of the vertices of G shown above. ■

Example 3.1B

Find a vertex colouring of the following graph G.



STEP 1 We label the vertices $a, \ldots, f$ as follows.



STEP 2 We colour vertex a with colour 1.

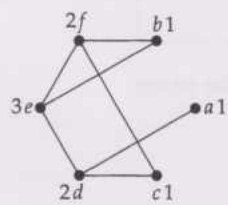
STEP 2 We colour vertex b with colour 1.

STEP 2 We colour vertex c with colour 1.

STEP 2 We colour vertex d with colour 2.

STEP 2 We colour vertex e with colour 3.

STEP 2 We colour vertex f with colour 2.



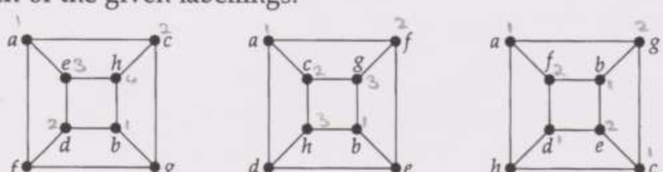
All the vertices are now coloured, so we STOP. We thus obtain the 3-colouring of the vertices of G shown above. ■

Notice that, in the above examples, $\chi(G) = 3$, and in Example 3.1B we found a vertex colouring of G which uses 3 colours.

See Problem 3.6, graph (d) for the value of $\chi(G)$.

Problem 3.7

Use the greedy algorithm to colour the vertices of the following graph G, using each of the given labellings.



(a) (b) (c)

$\chi(G) = 2$

What is the actual value of $\chi(G)$?

Example 3.1B and Problem 3.7 illustrate the following theorem.

Theorem 3.6

For any graph G, there is a labelling of the vertices for which the greedy algorithm yields a vertex colouring with $\chi(G)$ colours.

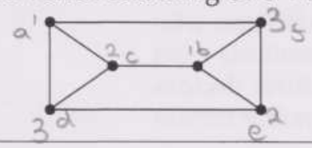
Outline of proof

Take any vertex colouring of G with $\chi(G)$ colours, denoted by 1, 2, 3, …, and sequentially label with $a, b, c, \ldots$ the vertices coloured 1, then the vertices coloured 2, then the vertices coloured 3, and so on. For this labelling, the greedy algorithm assigns the colours 1, 2, 3, … in that order, and so only $\chi(G)$ colours are needed. ■

Other labellings are possible.

Problem 3.8

Find a labelling of the vertices of the following graph, for which the greedy algorithm yields a vertex colouring of G with $\chi(G)$ colours.



We conclude this subsection by returning to a result that we proved earlier using the method of mathematical induction. We now outline an alternative proof using the greedy algorithm.

Theorem 3.1

Let G be a simple graph whose maximum vertex degree is d. Then

$\chi(G) \leq d + 1$.

Outline of proof

Let $a, b, c, \ldots$ be any labelling of the vertices of G. Colour these vertices in turn, using the lowest numbered colour available. At each stage the vertex to be coloured has at most d adjacent vertices, and there are $d + 1$ colours, so there is always a colour available. The colouring with $d + 1$ colours can therefore be completed. ■

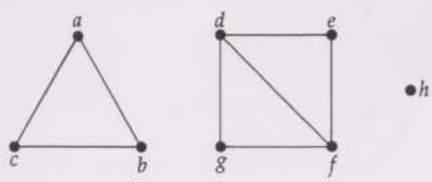
Using a more complicated version of the greedy algorithm, it is possible to prove Brooks' theorem (Theorem 3.2) in a similar way.

3.3 Vertex decompositions

Some of the most interesting problems in graph theory involve the decomposition of a graph G into subgraphs of a particular type. In several of these problems, we split the set of *vertices* of G into disjoint subsets; this is called a **vertex decomposition** of G.

Recall that subsets are *disjoint* if they have no element in common.

For example, consider the following disconnected graph G.



A natural vertex decomposition is to split the set of vertices into the disjoint subsets corresponding to the components of G:

$\{a, b, c\}, \{d, e, f, g\}, \{h\}$.

In this subsection, we adopt a similar approach to several other problems. Each of these problems can be formulated in graph-theoretic terms, and involves splitting the set of vertices of a graph into disjoint subsets with particular properties. By doing this, we can observe similarities between

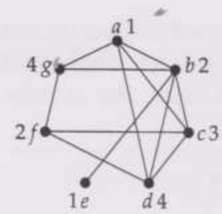
seemingly different problems, and begin to classify them, thereby gaining insight into the nature of the different types of problem.

We consider three types of problem — *colouring problems*, such as the chemical storage problem, the map colouring problem and problems involving tour graphs, and *domination* and *independence problems*, such as recreational problems involving queens on a chessboard.

Colouring problems

Example 3.1: storing chemicals

In Section 3.1 we considered the problem of a chemical manufacturer who wishes to store chemicals $a, b, \ldots, g$ in a warehouse. Some chemicals react violently when in contact with each other, and the manufacturer decides to divide the warehouse into a number of areas so as to separate certain pairs of chemicals.



In order to determine the smallest number of areas needed to store these chemicals safely, we drew the graph shown in the margin. The vertices correspond to the chemicals, and two vertices are joined by an edge whenever the corresponding chemicals must be kept separate; the numbers refer to the storage areas.

We saw that the assignment of chemicals to areas is an instance of a vertex colouring problem in which the colours correspond to the areas. Such a colouring gives rise to a vertex decomposition of the graph in which *no two vertices in the same subset are adjacent*. The vertex decomposition arising from this example is

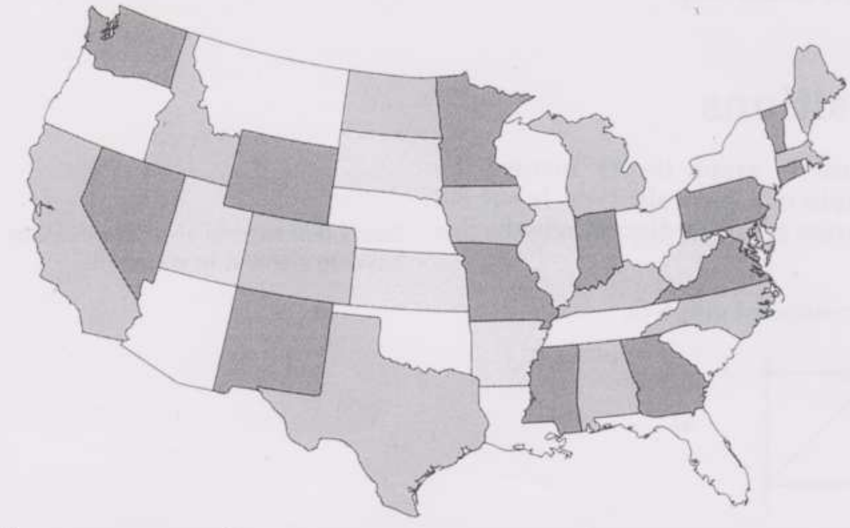
$\{a, e\}, \{b, f\}, \{c\}, \{d, g\}$;

the four subsets correspond to the chemicals in the four areas. In such a problem, the minimum number of subsets needed is simply the *chromatic number* of the corresponding graph. ■

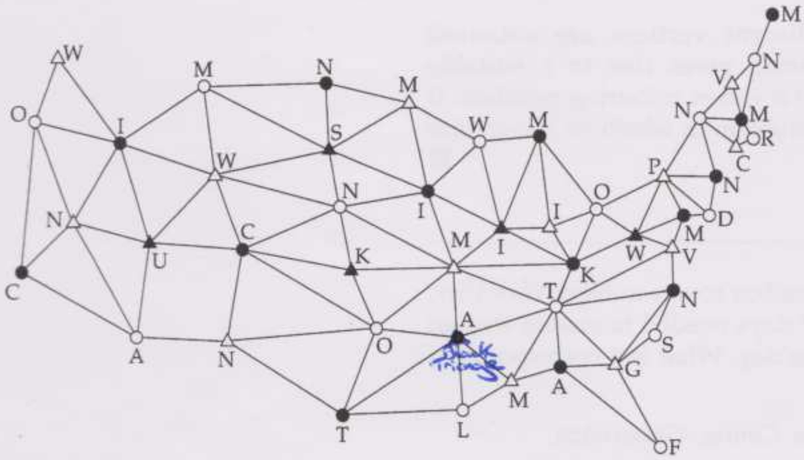
Example 3.2: map colouring

Recall that the map of the United States of America (excluding Alaska and Hawaii) can be coloured with just four colours, as follows.

Introduction unit, Example 2.4.



We can represent this as a vertex decomposition problem by considering the dual problem in which each state is represented by a vertex, and two vertices are joined whenever the corresponding states share a common boundary. This gives the following graph, in which each vertex has been assigned a symbol to represent the colour of the corresponding state. Since any two neighbouring states in the original map were coloured differently, any two adjacent vertices in this graph must also be assigned different colours.



The vertices corresponding to two regions whose common boundary is a single point, such as Colorado and Arizona, are *not* joined by an edge.

△ Washington, Nevada, Wyoming, New Mexico, Minnesota, Mississippi, Missouri, Indiana, Georgia, Virginia, Pennsylvania, Connecticut, Vermont;

○ Oregon, Montana, Arizona, Nebraska, Oklahoma, Louisiana, Wisconsin, Tennessee, Ohio, Florida, South Carolina, Delaware, New York, Rhode Island, New Hampshire;

● California, Idaho, Colorado, North Dakota, Texas, Iowa, Michigan, Alabama, Kentucky, North Carolina, Maryland, New Jersey, Massachusetts, Maine, ~~Arkansas~~;

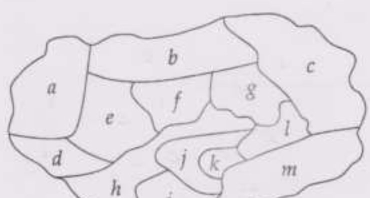
▲ Utah, South Dakota, Kansas, Illinois, West Virginia, Arkansas

Such a colouring of the vertices of the graph splits the set of vertices into four disjoint subsets, corresponding to the four colours.

This vertex decomposition has the property that *no two vertices in the same subset are adjacent*. ■

Problem 3.9

Consider the following map.



(a) Find a 4-colouring of this map by trial and error.

(b) Draw the corresponding graph, and show how the 4-colouring in part (a) leads to a vertex decomposition of this graph in which no two vertices in the same subset are adjacent.

Vertex decomposition problems also arise in situations which involve planning a tour, such as refuse collection.

Example 3.3: refuse collection

A weekly route schedule for refuse collection lorries is to be organized. The daily routes must be different for Monday to Saturday, and some sites need to be visited several times a week. No route is to be too long or too short, every lorry must be used on every working day, and every site must be visited the required number of times. How can a suitable schedule be designed?

In its full complexity, this problem is too hard to be considered here, so we look at just one aspect of it. We shall investigate whether it is possible to arrange a schedule in such a way that two different lorries do not visit the same site on the same day. To this end, we construct a *tour graph* in which each vertex represents a route, and two vertices are joined by an edge whenever the corresponding routes have a site in common. If the vertices of this tour graph can be coloured with six colours (corresponding to the

days Monday to Saturday) so that adjacent vertices are coloured differently, then any such vertex colouring gives rise to a suitable schedule. So the problem reduces to that of a vertex colouring problem. It is therefore again a vertex decomposition problem in which *no two vertices in the same subset are adjacent*. ■

Problem 3.10

Draw the tour graph for the following tourist bus routes in New York City, and use it to find the minimum number of days needed to ensure that no place is visited more than once on the same day. What is a corresponding vertex decomposition?

route 1 Empire State Building, Rockefeller Center, Greenwich Village, Pier 42

route 2 Rockefeller Center, Lincoln Center, Central Park, Columbia University

route 3 Madison Square Gardens, Rockefeller Center, United Nations

route 4 Metropolitan Art Museum, Central Park, Rockefeller Center

route 5 Metropolitan Art Museum, Columbia University, Lincoln Center

route 6 Columbia University, Bronx Zoo, Yankee Stadium

route 7 Shea Stadium, Yankee Stadium, Brooklyn Botanical Gardens

route 8 Bronx Zoo, Brooklyn Botanical Gardens

route 9 Empire State Building, Madison Square Gardens, Pier 42, United Nations

route 10 Pier 42, Statue of Liberty

route 11 Statue of Liberty, Wall Street, Greenwich Village

route 12 Wall Street, Greenwich Village, City College

Domination problems

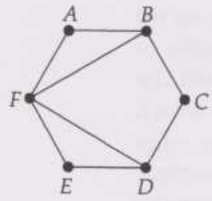
Communication links

Suppose that communication links are to be set up between a number of cities, and transmitting stations are to be built in some of these cities so that each city can receive messages from at least one transmitting station. For reasons of economy, we require the number of transmitting stations to be as small as possible. How can this be done?

We can represent this situation by a graph whose vertices correspond to the cities, and whose edges correspond to pairs of cities that can communicate with each other. Since each city must either contain a transmitting station or communicate with a city containing a transmitting station, we wish to find a set of vertices that (between them) are adjacent to all other vertices of the graph.

Example 3.4: locating transmitting stations

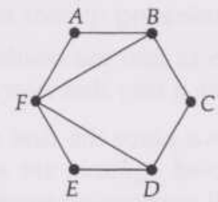
Suppose that the following graph represents the communication links between six cities, $A, \ldots, F$.



We can locate the transmitting stations at A, C and E, since each of the other vertices (B, D and F) is adjacent to at least one of these vertices; we say that the vertices A, C and E form a *dominating set*. We thus obtain a vertex decomposition into subsets of cities served by the same transmitting station:

$\{A, B, F\}, \{C, D\}, \{E\}$.

We omit B from the second subset, and D and F from the third subset, as they have already appeared in earlier subsets.



Note that we obtain a more economical solution by taking just two transmitting stations and locating them at A and D. As before, each of the other vertices (B, C, E and F) is adjacent to at least one of these vertices. Thus the vertices A and D form a *dominating set* that is smaller than the one we had above. A corresponding vertex decomposition is

$\{A, B, F\}, \{D, C, E\}$.

Notice that, in each of the above vertex decompositions, *each subset contains a vertex adjacent to all the other vertices in that subset.* ■

The location of transmitting stations in the above example illustrates the following definitions.

Definitions

A **dominating set** of vertices in a graph G is a set S of vertices with the property that each vertex of G is either in S or adjacent to a vertex of S.

A **minimum dominating set** is a dominating set of smallest possible size.

The **dominating number** of G, denoted by **dom (G)**, is the number of vertices in a minimum dominating set.

For example, in the above graph, the sets $S = \{A, C, E\}$ and $S = \{A, D\}$ are both dominating sets. Since there are no dominating sets of size 1, the set $S = \{A, D\}$ is a minimum dominating set, and the dominating number is 2.

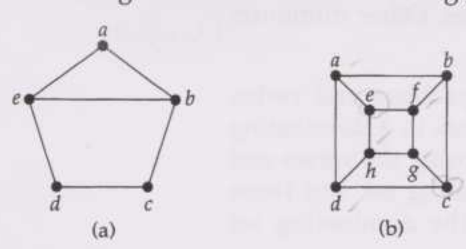
Problem 3.11

Find other minimum dominating sets in the above graph.

It follows from our discussion that the above communication problem reduces to that of finding a minimum dominating set in the corresponding graph. Such problems occur in many different guises. For example, suppose that a number of locations in a nuclear power plant are fitted with warning lights, and that sensors are to be stationed in various places to keep watch on these lights. We can minimize the number of sensors by finding a minimum dominating set in the corresponding graph and positioning the sensors accordingly. Any light that comes on can then be seen by at least one sensor, and appropriate action can be taken.

Problem 3.12

Find a minimum dominating set in each of the following graphs.



Problem 3.13

Write down dom (G), when G is:

(a) the complete bipartite graph $K_{3,4}$;

(b) the graph of the octahedron.

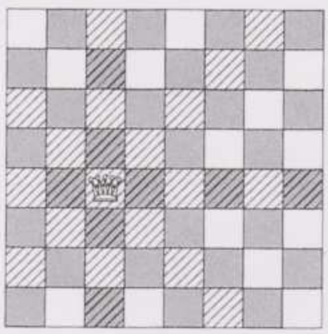
A recreational problem of this type is the following.

Example 3.5: dominating queens on a chessboard

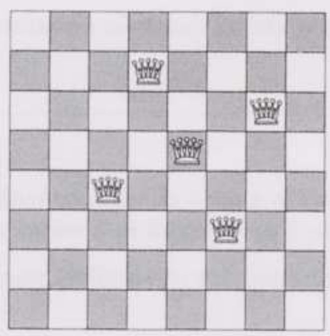
Suppose we wish to find the *smallest* set of queens that can be placed on a chessboard in such a way that every unoccupied square is attacked.

In chess, a queen attacks all squares in the same row or column and all squares in either diagonal through the square on which it is placed.

For example, if we place the first queen as shown in diagram (a) below, then 25 unoccupied squares are attacked. How many more queens are needed so that all unoccupied squares are attacked?



(a)



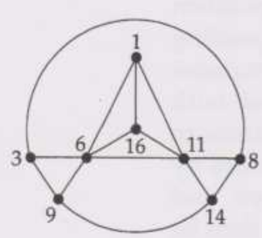
(b)

In fact, only four more queens are needed. An arrangement of five queens that attack all unoccupied squares is shown in diagram (b); it can be shown that no arrangement of four queens will do.

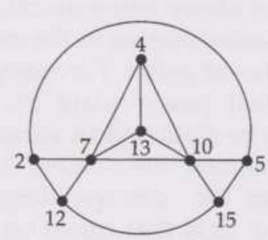
It can also be shown that five queens are sufficient for chessboards of size 9×9, 10×10 and 11×11.

We can represent this problem graphically by taking the squares of the chessboard as vertices, and joining two vertices by an edge whenever a queen can move from one square to the other. A solution to the problem then corresponds to finding a dominating set with 5 vertices, and showing that it is a minimum dominating set. Since the graph corresponding to an 8×8 chessboard has 64 vertices and 728 edges, we shall not attempt to draw it, but look instead at the analogous problem of a bishop on a 4×4 chessboard.

A bishop attacks all squares in either diagonal through the square on which it is placed.



white squares



black squares

In this case, the graph can be split into two parts, corresponding to the black squares and the white squares, respectively. There are several minimum dominating sets — for example, {6, 7, 10, 11}, which corresponds to placing a bishop on each of the central four squares. Other minimum dominating sets are {5, 6, 7, 8} and {9, 10, 11, 12}.

We can see the connection between this chessboard problem and vertex decomposition problems as follows. We list the vertices in a dominating set; then, for each such vertex, we form a subset comprising the vertex and its neighbours, omitting other vertices in the dominating set and those that have already been recorded. In this example, the dominating set {6, 7, 10, 11} gives the following subsets.

vertex 6 Its neighbours are vertices 1, 3, 9, 11 and 16; we omit vertex 11, since it lies in the dominating set, giving the subset {6, 1, 3, 9, 16}.

vertex 7 Its neighbours are vertices 2, 4, 10, 12 and 13; we omit vertex 10, since it lies in the dominating set, giving the subset {7, 2, 4, 12, 13}.

vertex 10 Its neighbours are vertices 4, 5, 7, 13 and 15; we omit vertices 4, 7 and 13, since they have already been recorded, giving the subset {10, 5, 15}.

vertex 11 Its neighbours are vertices 1, 6, 8, 14 and 16; we omit vertices 1, 6 and 16, since they have already been recorded, giving the subset {11, 8, 14}.

This gives the vertex decomposition

{6, 1, 3, 9, 16}, {7, 2, 4, 12, 13}, {10, 5, 15}, {11, 8, 14}. ■

Note that this type of vertex decomposition is very different from that produced in our discussion of colouring problems. For colouring problems, *in each subset, no two vertices are adjacent*. For domination problems, *each subset contains a vertex adjacent to all the other vertices in that subset.*

Problem 3.14

(a) For the minimum dominating set $\{a, c\}$ in Problem 3.12(a), write down a vertex decomposition with the above property.

(b) For the minimum dominating set $\{a, g\}$ in Problem 3.12(b), write down a vertex decomposition with the above property.

Problem 3.15

Draw the graph corresponding to knights' moves from each square on a 3×3 chessboard, and find a minimum dominating set and the corresponding vertex decomposition. Hence find the smallest number of knights that can be placed on such a chessboard in such a way that each unoccupied square is attacked.

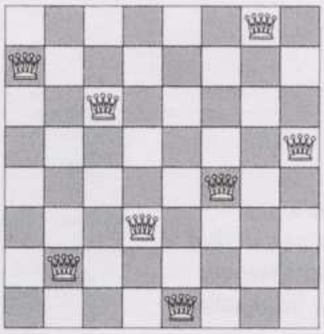
A knight attacks by moving two squares in one direction (horizontally or vertically), followed by one square in a perpendicular direction.

Independence problems

A related type of problem, known as an **independence problem**, is the following.

Example 3.6: independent queens on a chessboard

Suppose that we wish to find the *largest* set of queens that can be placed on a chessboard so that none of them is attacked by any other. Clearly, the number of queens cannot exceed 8, since at least two queens would then appear in the same row. On the other hand, it is certainly possible to place eight queens in the required manner, as shown in the following diagram.



The eight queens problem was studied by C. F. Gauss, who believed that there were 76 solutions. In 1854, the *Schachzeitung*, a Berlin chess journal, published 40 solutions. The correct number of solutions is 92.

As with the domination problem, we can represent this situation by a graph whose vertices correspond to the squares, and whose edges join pairs of squares that are connected by a queen's move. Again, we get a vertex decomposition of G into disjoint subsets, each of which contains a vertex adjacent to all the other vertices in that subset. ■

More generally, the *independence problem for a graph* G is that of finding the largest possible set of vertices of G, no two of which are adjacent.

As you saw above, this number is 8 for the chessboard example.

We formalize this idea, as follows.

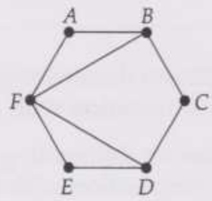
Definition

An **independent set** of vertices in a graph G is a set of vertices of G, no two of which are adjacent.

A **maximum independent set** is an independent set of largest possible size.

The **independence number** of G, denoted by **ind (G)**, is the number of vertices in a maximum independent set.

For example, for the graph in the margin, the sets $\{A, D\}$, $\{A, E\}$ and $\{A, C, E\}$ are all independent sets. The set $\{A, C, E\}$ is a maximum independent set and the independence number is 3.



Problem 3.16

Write down ind (G), when G is

(a) the complete bipartite graph $K_{3,4}$;

(b) the graph of the octahedron.

Finding dom (G) and ind (G)

In order to solve the domination and independence problems for a graph G, we need to find dom (G) and ind (G). For example, if G is the graph in the margin above, then

$$\text{dom}(G) = 2 \quad \text{and} \quad \text{ind}(G) = 3,$$

whereas if G is the graph for the 8×8 chessboard problem above, then

$$\text{dom}(G) = 5 \quad \text{and} \quad \text{ind}(G) = 8.$$

Unfortunately, there is no general formula that gives the values of dom (G) and ind (G) for a general graph G. However, the following theorem gives two inequalities involving dom (G) and ind (G).

Theorem 3.7

For any graph G with n vertices:

(a) $\text{dom}(G) \le \text{ind}(G)$;

(b) $\chi(G) \times \text{ind}(G) \ge n$.

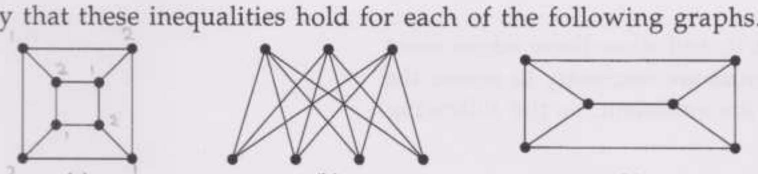
Proof

(a) Let S be a maximum independent set in G. Then S must be a dominating set, since otherwise there would be a vertex v in G that is not adjacent to any of the vertices in S; this vertex v could then be added to S to produce a larger independent set, which is impossible. The result follows.

(b) By the definition of $\chi(G)$, we can colour the vertices of G with $\chi(G)$ colours in such a way that no two adjacent vertices are assigned the same colour. It follows that the set of vertices of any given colour must form an independent set, and hence that there are at most ind (G) vertices of any given colour. Since there are $\chi(G)$ colours, the total number of vertices must be at most $\chi(G) \times$ ind (G). ■

Problem 3.17

Verify that these inequalities hold for each of the following graphs.



(a) (b) (c)

Hint For graph (b) use the results of Problems 3.3, 3.13 and 3.16; for graph (c) use the results of Problem 3.6.

3.4 Computer activities

The computer activities for this section are described in the *Computer Activities Booklet*.

After studying this section, you should be able to:

- explain the terms *vertex colouring, k-colouring* and *chromatic number*;
- state and use Brooks' theorem and the other theorems given in this section;
- apply the greedy algorithm for colouring the vertices of a graph;
- explain what are meant by a *colouring problem*, a *domination problem* and an *independence problem*.

4 Edge colourings and decompositions

In this section we consider problems involving the colouring of the *edges* of a graph.

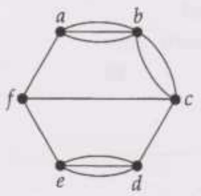
4.1 Edge colourings

Example 4.1: wire colouring

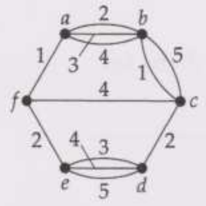
Suppose that we have a display panel on which electrical components a, b, ... are to be mounted and then interconnected. The connecting wires are first formed into a cable, with the wires to be connected to a emerging through one hole in the panel, those connected to b emerging through another hole, and so on. In order to distinguish all the wires that emerge from the same hole, they must be coloured differently. What is the minimum number of colours necessary for the whole system?

This problem was posed by C. E. Shannon in 1949, in a paper on electrical networks.

In order to investigate this problem, we represent the connection points by the vertices of a graph and the wires by edges. For example, the following graph represents a panel with six components, $a, \ldots, f$.



Since vertex b has five edges incident with it, and since these edges must all be coloured differently, at least five colours are necessary to colour the wires in this system. In fact, five colours are sufficient, as the following diagram shows.

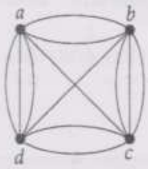


The numbers on the edges correspond to the five colours.

■

Problem 4.1

How many colours are needed to colour the edges of the graph in the margin so that any two edges incident with the same vertex are coloured differently?



The assignment of colours to wires in the above discussion illustrates the following definitions.

Definitions

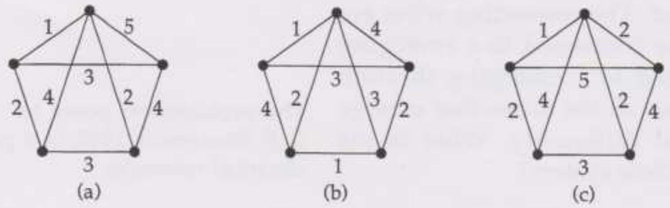
Let G be a graph without loops. A ***k*-edge colouring** of G is an assignment of k colours to the edges of G in such a way that any two edges meeting at a vertex are assigned different colours. If G has a k-edge colouring, then G is ***k*-edge colourable**.

The **chromatic index** of G, denoted by $\chi'(G)$, is the smallest number k for which G is k-edge colourable.

In the above wire colouring problem, the graph has chromatic index $\chi'(G) = 5$.

Note that the above definitions are given only for graphs *without loops*. Loops must be excluded since, in any k-edge colouring, the edges meeting at a vertex must be assigned different colours. However, we sometimes wish to consider graphs with multiple edges, since the introduction of multiple edges may alter the chromatic index, as in the wire colouring problem.

We usually show a k-edge colouring by writing the numbers $1, 2, \ldots, k$ next to the appropriate edges. For example, diagrams (a) and (b) below illustrate a 5-edge colouring and a 4-edge colouring of a graph G with eight edges; note that diagram (c) is *not* a 5-edge colouring of G, since two of the edges coloured 2 meet at a vertex.



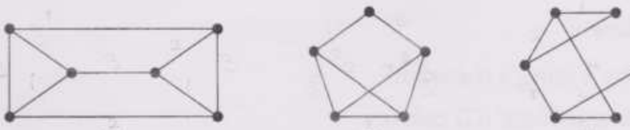
(a) (b) (c)

Since we have shown that G has a 4-edge colouring, it follows that $\chi'(G) \leq 4$. Also, G contains four edges meeting at a common vertex (a vertex of degree 4) which must be assigned different colours, so $\chi'(G) \geq 4$. Combining these inequalities, we obtain $\chi'(G) = 4$.

Thus 4 is an *upper bound* for $\chi'(G)$.
Thus 4 is a *lower bound* for $\chi'(G)$.

Problem 4.2

Determine $\chi'(G)$ for each of the following graphs G.



(a) (b) (c)

Hint For each graph, you need to devise a suitable edge colouring *and* explain why there is no edge colouring with fewer colours.

Problem 4.3

What can you say about the graphs G for which

(a) $\chi'(G) = 1$? (b) $\chi'(G) = 2$?

Problem 4.4

Decide whether each of the following statements about a graph G is TRUE or FALSE, and give a proof or counter-example as appropriate.

(a) If G contains a vertex of degree r, then $\chi'(G) \geq r$.

(b) If $\chi'(G) \geq r$, then G contains a vertex of degree r.

Given a particular graph G, how can we determine its chromatic index? We have seen that an upper bound for $\chi'(G)$ may be obtained by construction:

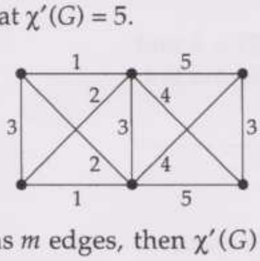
> to obtain an upper bound for $\chi'(G)$, construct an explicit colouring for the edges of G.

A lower bound for $\chi'(G)$ may be obtained using the result of Problem 4.4(a):

> to obtain a lower bound for $\chi'(G)$, find the maximum vertex degree in G.

For example, if G contains a vertex of degree 3, then $\chi'(G) \geq 3$.

If we can find an upper bound and a lower bound which are the same, then $\chi'(G)$ is equal to this common value. For example, the edges of the graph G below can be coloured with five colours, as shown, and so $\chi'(G) \leq 5$. But G cannot be coloured with fewer than 5 colours, since G contains a vertex of degree 5, and so $\chi'(G) \geq 5$. Combining these two inequalities, we deduce that $\chi'(G) = 5$.



Note that if a graph G has m edges, then $\chi'(G) \leq m$. However, this upper bound is usually rather poor.

This inequality becomes an equality ($\chi'(G) = m$) when G is a complete bipartite graph of the form $K_{1,m}$.

Much better upper bounds have been established by V. G. Vizing and by C. E. Shannon. For simple graphs, Vizing proved the following result in 1963, which we state without proof.

Theorem 4.1: Vizing's theorem

Let G be a simple graph whose maximum vertex degree is d. Then

$d \leq \chi'(G) \leq d + 1$.

This remarkable result tells us that, if G is any simple graph with maximum vertex degree d, then the chromatic index of G is either d or $d+1$. This classifies simple graphs into two classes: those for which $\chi'(G) = d$, and those for which $\chi'(G) = d + 1$. The graphs in Problem 4.2 show that both possibilities occur, but it is not known in general which graphs belong to which class.

Problem 4.5

For each of the following simple graphs G, write down:

the lower and upper bounds for $\chi'(G)$ given by Vizing's theorem;

the actual value of $\chi'(G)$, and an edge colouring using $\chi'(G)$ colours:

(a) the cycle graph C_7;

(b) the complete bipartite graph $K_{2,4}$;

(c) the complete graph K_6.

Before investigating the problem of classifying simple graphs into those with $\chi'(G) = d$ and those with $\chi'(G) = d + 1$, we state (without proof) two results that give upper bounds for the chromatic index of a graph with multiple edges. The first of these is an extension of Vizing's theorem.

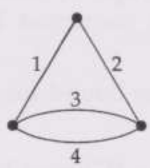
Theorem 4.2: Vizing's theorem (extended version)

Let G be a graph whose maximum vertex degree is d, and let h be the maximum number of edges joining a pair of vertices. Then

$d \le \chi'(G) \le d + h.$

This theorem reduces to the earlier version of Vizing's theorem when G is a simple graph.

For example, if G is the graph in the margin, then $d = 3$ and $h = 2$, since there are two edges joining a pair of vertices, and so the lower bound is 3 and the upper bound is 5; in fact, $\chi'(G) = 4$ for this particular graph.



Another upper bound for the chromatic index of a graph was obtained by Shannon in his paper on the wire colouring problem.

Theorem 4.3: Shannon's theorem

Let G be a graph whose maximum vertex degree is d. Then

$d \le \chi'(G) \le 3d/2,$ if d is even;

$d \le \chi'(G) \le (3d-1)/2,$ if d is odd.

For example, if G is the graph in the margin above, with $\chi'(G) = 4$ and $d = 3$, then $(3d - 1)/2 = 4$. So the lower bound is 3 and the upper bound is 4.

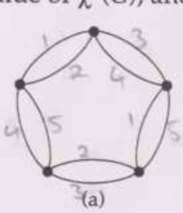
Problem 4.6

For each of the following graphs G, write down:

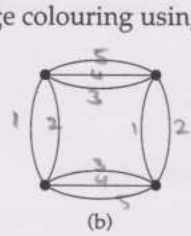
the lower and upper bounds for $\chi'(G)$ given by Vizing's theorem (extended version);

the lower and upper bounds for $\chi'(G)$ given by Shannon's theorem;

the actual value of $\chi'(G)$, and an edge colouring using $\chi'(G)$ colours.



(a)



(b)

We summarize the above results as follows.

To find the chromatic index $\chi'(G)$ of a graph G without loops

Try to find an upper bound and a lower bound which are the same; then $\chi'(G)$ is equal to this common value.

If $\chi'(G) \leq k$ and $\chi'(G) \geq k$, then $\chi'(G) = k$.

possible upper bounds for $\chi'(G)$

- the number of colours in an explicit edge colouring of G;
- the number m of edges in G;
- $d + 1$, where d is the maximum vertex degree in G, provided that G has no multiple edges; (Vizing's theorem)
- $d + h$, where d is the maximum vertex degree in G and h is the maximum number of edges joining a pair of vertices; (Vizing's theorem (extended version))
- $3d/2$, where d is the maximum vertex degree and d is even; (Shannon's theorem)
- $(3d - 1)/2$, where d is the maximum vertex degree and d is odd. (Shannon's theorem)

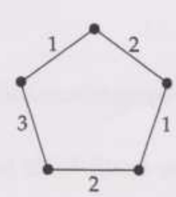
possible lower bound for $\chi'(G)$

- d, the maximum vertex degree in G.

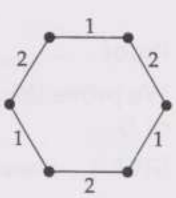
Classifying simple graphs

We now return to the problem of classifying simple graphs into two classes: those with $\chi'(G) = d$ and those with $\chi'(G) = d + 1$. For some types of graph, this is very straightforward; for example, it is easy to show that, for the cycle graphs C_n ($n \geq 3$), we have

$$\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ is even;} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$



$n = 5$



$n = 6$

A similar result holds for the complete graph K_n.

Theorem 4.4

For the complete graph K_n,

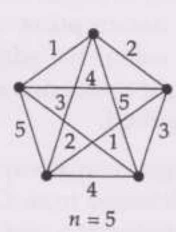
$$\chi'(K_n) = \begin{cases} n - 1 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Proof

Since each vertex has degree $n - 1$, it follows from Vizing's theorem that $\chi'(K_n)$ is either $n - 1$ or n.

If n is odd, then the maximum number of edges that can be assigned the same colour is $(n - 1)/2$, since otherwise two of these edges meet at a common vertex. But K_n has exactly $n(n - 1)/2$ edges, and so the number of colours must be at least n. Hence $\chi'(K_n) \geq n$.

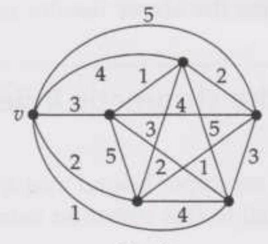
In fact, we can obtain an explicit n-edge colouring of K_n by drawing the vertices in the form of a regular n-gon, and colouring the edges of the boundary using a different colour for each edge. Each of the remaining edges is then assigned the same colour as the boundary edge parallel to it. It follows that $\chi'(K_n) \leq n$.



$n = 5$

Combining the above inequalities, we deduce that $\chi'(K_n) = n$, if n is odd.

If n *is even*, we can prove that $\chi'(K_n) = n - 1$, by explicitly constructing an $(n-1)$-edge colouring of K_n. If $n = 2$, this is trivial. If $n > 2$, we choose any vertex v and remove it, together with its incident edges. This leaves a complete graph K_{n-1} with an odd number of vertices, whose edges can be coloured with $n - 1$ colours, using the above construction. At each vertex there is exactly one colour missing, and these missing colours are all different. The edges of K_n incident to v can therefore be coloured using these missing colours. It follows that $\chi'(K_n) = n - 1$, if n is even. ■



$n = 6$

Problem 4.7

(a) Suppose that 31 teams take part in a competition in which each team must play exactly one match against each of the other 30 teams. If no team can play more than one match a day, how many days are needed?

(b) What is the corresponding answer if there are 32 teams, each of which must play exactly one match against each of the other 31 teams?

We conclude this subsection with a theorem of Dénes König, which tells us that the edges of any bipartite graph (not necessarily simple) with maximum vertex degree d can be coloured with just d colours.

Dénes König was a Hungarian mathematician who wrote the first comprehensive treatise on graph theory, *Theorie der Endlichen und Unendlichen Graphen* (Theory of Finite and Infinite Graphs), in 1936.

Theorem 4.5: König's theorem

Let G be a bipartite graph whose maximum vertex degree is d. Then

$\chi'(G) = d$.

Proof

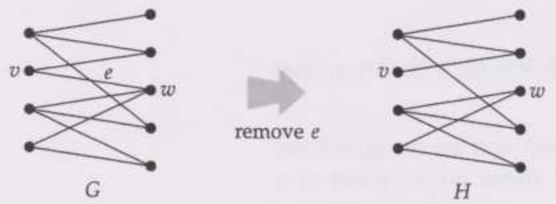
We prove this result by mathematical induction on m, the number of edges of G.

STEP 1 *Show that the result holds for the graph with one edge.*

When $m = 1$, we have $\chi'(G) = 1$ and $d = 1$, and so the result holds in this case.

STEP 2 *Show that, for each positive integer m, if the result holds for all bipartite graphs with fewer than m edges, then it must also hold for all bipartite graphs with m edges.*

Suppose that the result is true for all bipartite graphs with fewer than m edges. Let G be a bipartite graph with m edges and maximum vertex degree d, and let H be the graph obtained from G by removing an edge e adjacent to the vertices v and w:



Since H has fewer than m edges and maximum vertex degree d (or less), it follows from our assumption that $\chi'(H) \leq d$; that is, H is d-edge colourable. We now colour the edges of H with d colours, and replace the edge e. If we can colour e with one of the d colours, then we obtain a d-edge colouring of G, as required.

To show that the edge e can always be coloured in this way, we argue as follows. Since H is obtained from G by removing the edge e, there must be at least one colour missing at v, and at least one colour missing at w.

If there is some colour missing at *both* v and w, then we can assign this colour to the edge e, thereby completing the d-edge colouring of G.

If this is not the case, suppose that the colour blue is missing at v, and the colour red is missing at w, and consider the path starting at v and consisting entirely of red and blue edges. The edges in such a path must alternate in colour, and must alternate between the vertices on the left and those on the right of the bipartite graph. Since there are no red edges at w, the colour red must appear at v. It follows that w *cannot* be reached from v by such a red–blue path, since w would have to be reached by a red edge.

interchange red and blue

We now interchange the colours on this path, so that the blue edges become red, and the red edges become blue. Then the colours appearing at w are unchanged, and the colour red is now missing at both v and w. We can therefore assign to the edge e the colour red, thereby completing the colouring of the edges of G.

This is where we use the fact that G, and hence H, is bipartite.

It follows that the result holds for all bipartite graphs with m edges. This completes Step 2.

Therefore, by the principle of mathematical induction, the result holds for all bipartite graphs with m edges, for each positive integer m. It therefore holds for all bipartite graphs. ■

Problem 4.8

Use König's theorem to write down the chromatic index of each of the following graphs:

(a) the complete bipartite graph $K_{r,s}$ ($r \leq s$);

(b) the graph of the cube;

(c) the k-cube Q_k.

4.2 Algorithm for edge colouring

In Section 3.2 we presented a greedy algorithm for colouring the vertices of a graph. This algorithm is easy to apply, and usually gives good answers in practice. We now present a corresponding algorithm for edge colouring.

Greedy algorithm for edge colouring

START with a graph G and list of colours 1, 2, 3, ….

STEP 1 Label the edges $a, b, c, \ldots$ in any manner.

STEP 2 Identify the uncoloured edge labelled with the earliest letter in the alphabet.

Colour it with the first colour in the list not used for any coloured edge that meets it at a vertex.

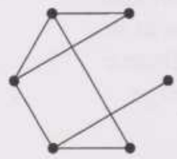
Repeat Step 2 until all the edges are coloured, then STOP.

An edge colouring of G has been obtained. The number of colours used depends on the labelling chosen for the edges in Step 1.

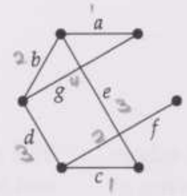
We illustrate the use of the algorithm by two examples using the same graph with different labellings.

Example 4.1A

Find an edge colouring of the following graph G.



STEP 1 We label the edges a, …, g as follows.



STEP 2 We colour edge a with colour 1.

STEP 2 We colour edge b with colour 2.

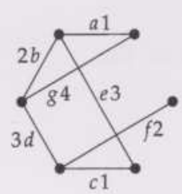
STEP 2 We colour edge c with colour 1.

STEP 2 We colour edge d with colour 3.

STEP 2 We colour edge e with colour 3.

STEP 2 We colour edge f with colour 2.

STEP 2 We colour edge g with colour 4.

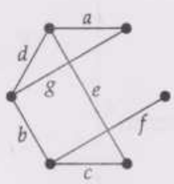


All the edges are now coloured, so we STOP. We thus obtain the 4-edge colouring of G shown above. ■

Example 4.1B

Find an edge colouring of the following graph G.

STEP 1 We label the edges a, …, g as follows.



STEP 2 We colour edge a with colour 1.

STEP 2 We colour edge b with colour 1.

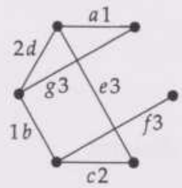
STEP 2 We colour edge c with colour 2.

STEP 2 We colour edge d with colour 2.

STEP 2 We colour edge e with colour 3.

STEP 2 We colour edge f with colour 3.

STEP 2 We colour edge g with colour 3.



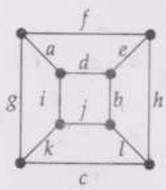
All the edges are now coloured, so we STOP. We thus obtain the 3-edge colouring of G shown above. ■

Notice that, in the above examples, $\chi'(G) = 3$ and in Example 4.1B we found an edge colouring of G which uses 3 colours.

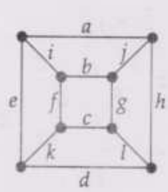
See Problem 4.2, graph (c) for the value of $\chi'(G)$.

Problem 4.9

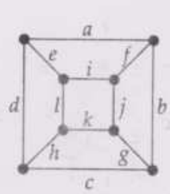
Use the greedy algorithm to colour the edges of the following graph G, using each of the given labellings.



(a)



(b)



(c)

What is the actual value of $\chi'(G)$?

Example 4.1B and Problem 4.9 illustrate the following theorem.

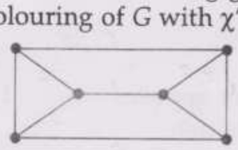
> **Theorem 4.6**
>
> For any graph G, there is a labelling of the edges for which the greedy algorithm yields an edge colouring with $\chi'(G)$ colours.

Outline of proof

Take any edge colouring of G with $\chi'(G)$ colours, denoted by 1, 2, 3, ... and sequentially label with a, b, c, ... the edges coloured 1, then the edges coloured 2, then the edges coloured 3, and so on. For this labelling, the greedy algorithm assigns the colours 1, 2, 3, ... in that order, and so only $\chi'(G)$ colours are needed. ■

Problem 4.10

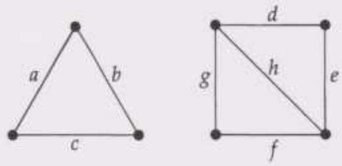
Find a labelling of the edges of the following graph, for which the greedy algorithm yields an edge colouring of G with $\chi'(G)$ colours.



4.3 Edge decompositions

Some of the most interesting problems in graph theory involve the decomposition of a graph G into subgraphs of a particular type. In several of these problems, we split the set of *edges* into disjoint subsets; this is called an **edge decomposition** of G.

For example, consider the following disconnected graph G.



A natural edge decomposition is to split the set of edges into disjoint subsets corresponding to the components of G:

$\{a, b, c\}, \{d, e, f, g, h\}$.

Another natural edge decomposition arises from an idea introduced earlier in the course. In *Graphs 1*, we introduced the idea of an Eulerian graph G, and we investigated conditions under which a given connected graph is Eulerian. In particular, we showed that every Eulerian graph can be split into disjoint cycles — this means that we can split the set of edges of G into disjoint subsets.

For example, if G is the Eulerian graph below, then there are five edge decompositions of G into disjoint cycles:

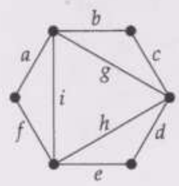
$\{a, b, c, d, e, f\}, \{g, h, i\}$;

$\{a, f, i\}, \{b, c, g\}, \{d, e, h\}$;

$\{a, f, h, g\}, \{b, c, d, e, i\}$;

$\{b, c, h, i\}, \{a, f, e, d, g\}$;

$\{d, e, i, g\}, \{a, b, c, h, f\}$.

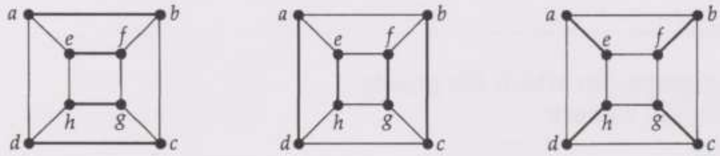


In this subsection, we adopt a similar approach to several other problems. Each of these problems can be formulated in graph-theoretic terms, and involves splitting the set of edges of a graph into disjoint subsets with particular properties. By doing this, we can observe similarities between seemingly different problems and begin to classify them, thereby gaining insight into the nature of the different types of problem.

We consider three types of problem — *matching problems*, such as wire colouring and scheduling examinations, *problems requiring decomposition of a graph into planar subgraphs* such as the printed circuits problem, and *problems requiring decomposition of a graph into spanning trees*, such as the allocation of bus routes between a number of towns.

Matching problems

The following diagram shows the cube graph and three sets of edges indicated by thick lines.



These three sets have the property that every edge of the graph appears in just one of them, and this leads to the following edge decomposition:

$\{ab, cd, ef, gh\}, \{ad, bc, eh, fg\}, \{ae, bf, cg, dh\}$.

Each of the above sets consists of edges which have no vertex in common. Such a set of edges of a graph is called a *matching*.

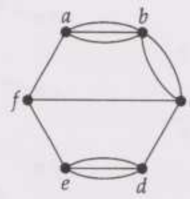
> **Definition**
>
> A **matching** in a graph G is a set of edges of G, no two of which have a vertex in common.

Every graph can be decomposed into matchings, since if there are m edges, then we can simply take m matchings, each consisting of a single edge. However, the problem of determining the *minimum* number of matchings needed to decompose a given graph may be much more difficult, and is unsolved in general. This question is of more than academic interest, and has arisen in several contexts, two of which we consider below.

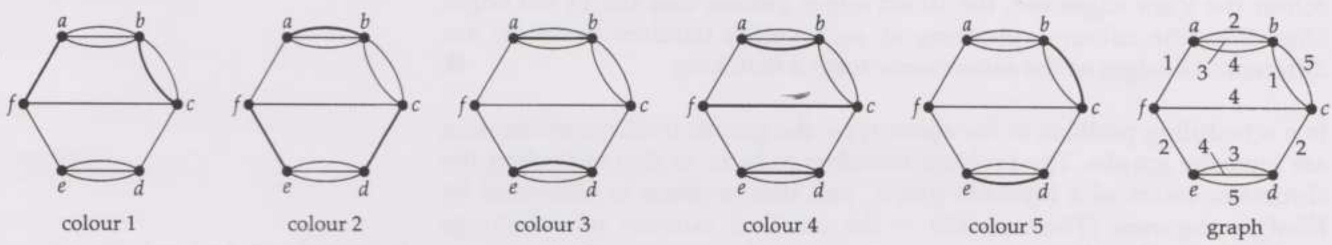
Notice that the problem of decomposing a graph into the minimum number of matchings is an edge colouring problem in which the edges of each matching are assigned the same colour.

Example 4.1: wire colouring

In Section 4.1 we considered a display panel on which six electrical components $a, \ldots, f$ are mounted and then interconnected. The connecting wires are first formed into a cable, with the wires to be connected to a emerging through one hole in the panel, those connected to b emerging through another hole, and so on. In order to distinguish them, all those wires that emerge from the same hole are coloured differently.



In order to determine the minimum number of colours necessary for the whole system, we represented the connection points by the vertices of a graph and the wires by edges. We found that five colours were necessary to colour the wires in the system. The following diagrams show the edges of each colour.



The edge decomposition corresponding to the above edge colouring consists of the edges coloured 1, the edges coloured 2, and so on. ■

In a wire colouring problem, the edges of each colour form a matching, so the problem of finding the smallest number of colours needed to colour the wires is the same as that of determining the minimum number of matchings needed to decompose the graph. Thus, *the wire colouring problem is an edge decomposition problem in which the edges in each subset form a matching.*

Since the graphs considered in wire colouring problems usually have multiple edges, the best we can say is that the number of matchings is limited by the bounds for the chromatic index given by the extended version of Vizing's theorem (Theorem 4.2) and Shannon's theorem (Theorem 4.3) — namely,

$$d \le \chi'(G) \le d + h \quad \text{and} \quad d \le \chi'(G) \le \tfrac{3}{2}d,$$

In Example 4.1, $d = 5$, $h = 3$ and $\chi'(G) = 5$.

where d is the maximum vertex degree in the graph G and h is the maximum number of edges joining a pair of vertices.

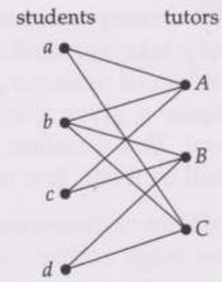
Since it is possible to find graphs attaining any of these bounds, we cannot obtain better results than this in general.

Example 4.2: scheduling examinations

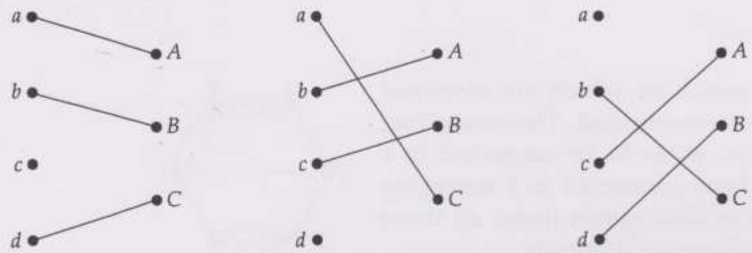
At the end of an academic year, all students have to take an hour-long examination with each of their tutors. How many examination periods are required?

We can see what is involved if we consider a simple example with four students a, b, c, d and three tutors A, B, C. We represent the students and

tutors by the vertices of a bipartite graph, and join a student vertex to a tutor vertex whenever the student needs to be examined by the tutor. An example of such a graph is shown in the margin.



If two edges meet at a common vertex, then the corresponding examinations cannot take place simultaneously. So the problem is an edge decomposition problem in which we must split the graph into subgraphs in which no two edges meet — that is, into matchings. In this particular case, the minimum number of matchings is 3, and a suitable timetable is as follows.



9 am 10 am 11 am

The corresponding edge decomposition is

$\{aA,\ bB,\ dC\},\ \{aC,\ bA,\ cB\},\ \{bC,\ cA,\ dB\}.$

This problem can also be thought of as an edge colouring problem. If we colour the 9 am edges red, the 10 am edges yellow, and the 11 am edges blue, then the colours appearing at each vertex (student or tutor) are different. All edges of the same colour form a matching. ■

In a scheduling problem of the above type, the graphs under consideration are bipartite graphs. The problem therefore reduces to that of finding the chromatic index of a bipartite graph, and this problem is answered by König's theorem (Theorem 4.5) — the smallest number of matchings needed is equal to the maximum vertex degree d in the bipartite graph.

In Example 4.2, $d = 3$ and $\chi'(G) = 3$.

Problem 4.11

Five students $a, \ldots, e$, are to be examined by five tutors $A, \ldots, E$:

tutor A must examine students b and d;

tutor B must examine students a, b and e;

tutor C must examine students b, c and e;

tutor D must examine students a and c;

tutor E must examine students b, d and e.

If each examination takes the same amount of time, find the minimum number of examination periods needed, and devise a suitable schedule.

Decomposition into planar subgraphs

Printed circuits problem

Recall that in printed circuits, electronic components are connected by means of conducting strips printed directly onto a flat board of insulating material. Such printed connectors may not cross, since this would lead to undesirable electrical contact at crossing points.

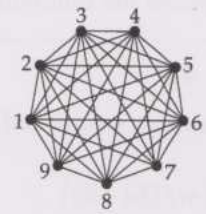
Circuits in which a large number of crossings are unavoidable may be printed on several boards which are then sandwiched together. Each board consists of a printed circuit without crossings. What is the smallest number of such layers needed for a given circuit?

We illustrate this problem with a particular example.

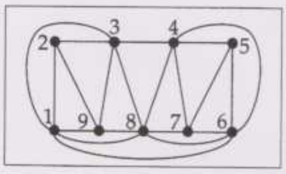
Example 4.3: printed circuits

Consider a printed circuit that has 36 interconnections and is represented by the complete graph K_9. Then it is impossible to arrange all these interconnections in one layer, or even two. Three layers are needed, and a solution is given below. Note that every edge of K_9 is included on one of the layers — for example, the edge 28 appears on layer 2, and the edge 69 appears on layer 3.

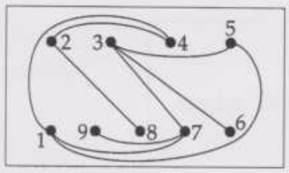
It can be shown that any drawing of K_9 in the plane must involve at least 36 crossings of two edges. The following drawing of K_9 has 126 crossings.



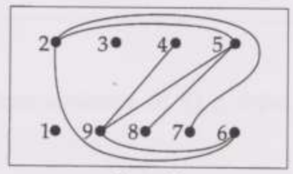
K_9



layer 1



layer 2



layer 3

Each of these three graphs is a planar graph. So the printed circuits problem reduces to the problem of decomposing the graph into smaller graphs, each of which is planar. In other words, it is an edge decomposition problem in which *the edges in each subset form a planar graph*. In the case of K_9, we get the following edge decomposition corresponding to the three layers shown above.

{12, 13, 16, 18, 19, 23, 29, 34, 38, 39, 45, 46, 47, 48, 56, 57, 67, 68, 78, 89}

{14, 15, 17, 24, 28, 35, 36, 37, 79}

{25, 26, 27, 49, 58, 59, 69} ■

Problem 4.12

Show that K_6 can be 'printed' in two layers, and write down a corresponding edge decomposition.

The above idea of splitting a graph into planar graphs leads to the following definition.

> **Definition**
>
> The **thickness** of a graph G, denoted by $t(G)$, is the minimum number of planar graphs that can be superimposed to form G.

For example, the thickness of any planar graph is 1, and the thickness of the complete graph K_9 is 3.

Problem 4.13

Determine the thickness of each of the following graphs:

(a) the complete graph K_5;

(b) the complete bipartite graph $K_{3,3}$;

(c) the Petersen graph.

In general, there is no known formula that gives the thickness of a graph G. However, we can easily obtain a *lower bound* for $t(G)$ that often coincides with the correct value. We restrict our attention to simple graphs, since we can collapse any multiple edges to a single edge and remove any loops, as we did in Section 1. We adopt the following notation.

Notation

Let a be any positive number. Then $\lfloor a \rfloor$ is the integer obtained by 'rounding a down', and $\lceil a \rceil$ is the integer obtained by 'rounding a up'.

The connection between these functions is given by

$$\lceil a/b \rceil = \lfloor a/b + (b-1)/b \rfloor.$$

For example, $\lfloor \pi \rfloor = 3$, $\lfloor 6.2 \rfloor = 6$ and $\lfloor 4 \rfloor = 4$; $\lceil \pi \rceil = 4$, $\lceil 6.2 \rceil = 7$ and $\lceil 4 \rceil = 4$.

For example, $\lceil 7/5 \rceil = \lfloor 7/5 + 4/5 \rfloor = \lfloor 11/5 \rfloor = 2$.

Note that, if a is an integer, then $\lfloor a \rfloor = \lceil a \rceil = a$.

We can now prove the following result.

Theorem 4.7

Let G be a simple connected graph with n (≥ 3) vertices and m edges. Then

(a) $t(G) \geq \lceil m/(3n-6) \rceil$;

(b) $t(G) \geq \lceil m/(2n-4) \rceil$, if G has no triangles.

Proof

(a) By Corollary 1.1, the number of edges in a simple connected planar graph with n (≥ 3) vertices and m edges is at most $3n-6$. Thus the number of edges on each 'layer' of G is at most $3n-6$. Since there are m edges altogether, the number of planar graphs must be at least $m/(3n-6)$. However, the number of planar graphs is an integer, and so $t(G) \geq \lceil m/(3n-6) \rceil$.

(b) By Corollary 1.2, the number of edges in a simple connected planar graph with n (≥ 3) vertices, m edges and no triangles is at most $2n-4$. Since there are m edges altogether, the number of planar graphs must be at least $m/(2n-4)$. However, the number of planar graphs is an integer, and so $t(G) \geq \lceil m/(2n-4) \rceil$. ■

We can now deduce lower bounds for the thickness of K_n and that of $K_{r,s}$.

Theorem 4.8

(a) $t(K_n) \geq \lfloor (n+7)/6 \rfloor$;

(b) $t(K_{r,s}) \geq \lceil rs/(2r+2s-4) \rceil$.

Proof

(a) If $G = K_n$, then $m = \frac{1}{2}n(n-1)$.

It follows from part (a) of Theorem 4.7 that

$$t(K_n) \geq \lceil \tfrac{1}{2}n(n-1)/(3n-6) \rceil.$$

We can rewrite this as follows:

$$\begin{aligned}
&\lceil \tfrac{1}{2}n(n-1)/(3n-6) \rceil \\
&= \lfloor \tfrac{1}{2}n(n-1)/(3n-6) + (3n-7)/(3n-6) \rfloor \\
&= \lfloor (\tfrac{1}{2}(n^2-n) + (3n-7))/(3n-6) \rfloor \\
&= \lfloor ((n^2-n) + (6n-14))/2(3n-6) \rfloor \\
&= \lfloor (n^2+5n-14)/2(3n-6) \rfloor \\
&= \lfloor (n+7)(n-2)/6(n-2) \rfloor \\
&= \lfloor (n+7)/6 \rfloor.
\end{aligned}$$

Thus

$$t(K_n) \geq \lfloor (n+7)/6 \rfloor.$$

It can be shown that $t(K_n) = \lfloor (n+7)/6 \rfloor$ for *all* n, except for $n = 9$ and $n = 10$, in which case $t(K_n) = 3$.

(b) If $G = K_{r,s}$, then $m = rs$ and G has no triangles.

It follows from part (b) of Theorem 4.7 that

$$t(K_{r,s}) \geq \lceil m/(2n-4) \rceil = \lceil rs/(2(r+s)-4) \rceil.$$

Thus

$$t(K_{r,s}) \geq \lceil rs/(2r+2s-4) \rceil.$$ ■

It is not known whether this inequality is always an equality, but it certainly is for all complete bipartite graphs with fewer than 48 vertices.

So, to sum up, although we cannot solve the printed circuits problem in general, we have obtained a lower bound for the solution, and this bound coincides with the correct value surprisingly often.

Problem 4.14

Use the above results to find the thickness of each of the following graphs.

(a) K_{20}; (b) $K_{20,20}$.

Decomposition into spanning subgraphs

A spanning subgraph of a graph G is a subgraph which contains all the vertices of G.

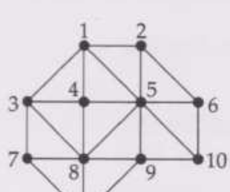
Bus route problems

In a certain county there are a number of rival bus companies. Each company wishes to run a service that includes every town in the county, in such a way that passengers using that company can get from any town to any other town. However, the County Council will not allow different companies to operate along the same stretch of road. How many different bus companies can be accommodated?

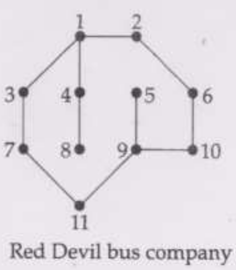
We can solve this problem by drawing a graph whose vertices correspond to the towns and whose edges correspond to the roads joining them.

Example 4.4: bus routes

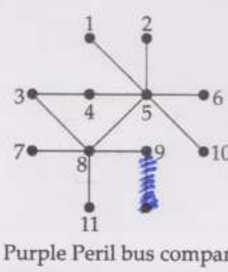
The following graph represents a county containing 11 towns joined by 22 roads.



Each bus company needs a network that connects all the 11 towns, and so each company must be assigned at least 10 of the interconnecting roads. Since there are 22 roads, the maximum number of companies that can be accommodated is 2. The following diagram shows an appropriate allocation of roads to the two companies.



Red Devil bus company



Purple Peril bus company ■

Such an allocation of roads to companies produces an edge decomposition of the original graph. Each subgraph in this decomposition must include

edges incident with all the vertices, and must be connected, so that a passenger can travel from any town to any other by the buses of each company. So the problem reduces to that of decomposing the graph into the maximum number of connected subgraphs, each of which includes every vertex of the graph; such subgraphs are called *spanning subgraphs*. We denote the number of subgraphs in such a decomposition by $s(G)$.

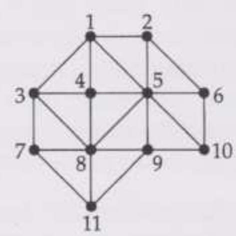
An expression for the number $s(G)$ was obtained by W. T. Tutte, who proved the following result in 1961.

Theorem 4.9

Let G be a connected graph with n vertices. Then $s(G)$ is the largest integer for which the following statement is true:

> for each positive integer $k \leq n$, at least $(k-1) \times s(G)$ edges must be removed in order to disconnect G into k components.

To illustrate this theorem, we consider the graph G in the margin, for which $s(G) = 2$, as we saw in Example 4.4.



In order to disconnect G into

2 components, we must remove at least 3 edges, so

$$s(G) \leq 3/(2-1) = 3;$$

3 components, we must remove at least 5 edges, so

$$s(G) \leq 5/(3-1) = 5/2;$$

4 components, we must remove at least 7 edges, so

$$s(G) \leq 7/(4-1) = 7/3;$$

...

11 components, we must remove all 22 edges, so

$$s(G) \leq 22/(11-1) = 22/10.$$

The largest integer $s(G)$ which satisfies all these inequalities is 2, as required.

The formal proof of Theorem 4.9 is too complicated to include here, but the following remarks indicate why the condition is necessary.

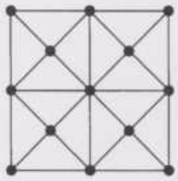
Assume that the graph G has been disconnected into k components by the removal of r edges. In order to have a connected system, each bus company must have at least $k-1$ linking edges between the various components. Thus, if there are $s(G)$ bus companies, we must have

$$r \geq (k-1) \times s(G),$$

as required.

Problem 4.15

Find the value of $s(G)$ for the following road network G.



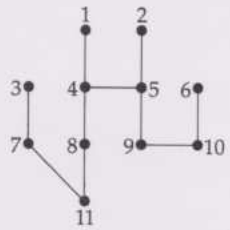
Several variations of the above problem lead to interesting mathematical results. For example, suppose that each bus company operates from a depot in one of the towns and chooses each of its routes to be a path out to another vertex, returning the same way. Then *each of the connected*

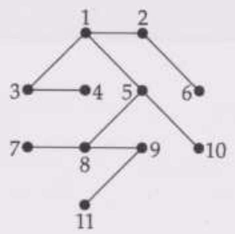
subgraphs must be a tree — in other words, the graph can be decomposed into spanning trees.

Such a decomposition is possible only if the number of edges in the graph is a multiple of the number of edges in a spanning tree; if the graph has n vertices and m edges, then m must be a multiple of $n - 1$.

Example 4.5: bus routes

In the above example, where $n = 11$ and $m = 22$, this can be accomplished only if two of the roads are not used by either company. For example, if the roads 38 and 56 are removed from the graph, then the resulting graph can be decomposed into the following spanning trees.

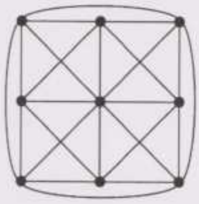




■

Problem 4.16

Decompose the following graph into disjoint spanning trees.



The following theorem gives a necessary and sufficient condition for the existence of a solution to this problem.

> **Theorem 4.10**
>
> Let G be a connected graph with n vertices and $s(n - 1)$ edges. Then G can be decomposed into s spanning trees if and only if
>
> for each positive integer $k \leq n$, at least $(k - 1) \times s$ edges must be removed in order to disconnect G into k components.

Proof

By Theorem 4.9, this theorem asserts that G can be decomposed into s spanning trees if and only if $s = s(G)$. However, if G can be decomposed into s connected subgraphs each of which includes every vertex of the graph, then each such subgraph must have $n - 1$ edges, and must therefore be a spanning tree, since there are no edges left to form any cycles. ■

We have now found an expression for the maximum number of bus companies that can be accommodated in the first type of problem, and we have obtained a necessary and sufficient condition for the existence of a solution to the second type of problem.

4.4 Computer activities

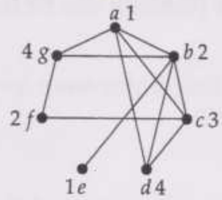
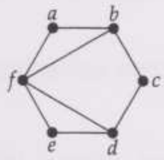
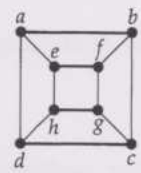
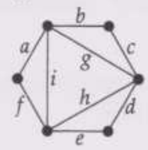
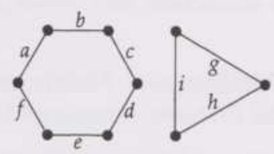
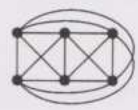
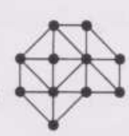
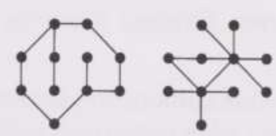
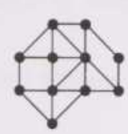
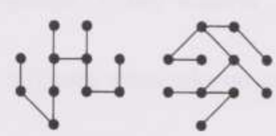
The computer activities for this section are described in the *Computer Activities Booklet*.

After studying this section, you should be able to:

- explain the terms *edge colouring, k-edge colouring, chromatic index* and *edge decomposition*;
- use Vizing's theorem (both versions), Shannon's theorem, König's theorem and the other theorems given in this section;
- apply the greedy algorithm for colouring the edges of a graph;
- explain what are meant by *edge colouring problems, matching problems*, the *printed circuits problem*, and various *bus route problems*, and show how each can be represented as an edge decomposition problem.

Summary

We conclude with a table showing some of the different types of decomposition problem described in Sections 3 and 4.

problem	type of decomposition	typical graph	decomposition
vertex colouring problems (storing chemicals, map colouring, refuse collection)	**vertex decomposition** (no two vertices in the same subset are adjacent)		$\{a, e\}$, $\{b, f\}$ $\{c\}$, $\{d, g\}$
domination problems (communication links, queens on a chessboard).	**vertex decomposition** (each subset contains a vertex adjacent to the other vertices in the subset)		$\{a, b, f\}$ $\{d, e, c\}$
matching problems (wire colouring, scheduling examinations)	**edge decomposition** into matchings (no two edges have a vertex in common)		$\{a, b\}$ $\{c, d\}$ $\{e, f\}$ $\{g, h\}$
Eulerian graph	**edge decomposition** into disjoint cycles		
printed circuits problem	**edge decomposition** into planar subgraphs		
bus-route problem	**edge decomposition** into connected subgraphs which include every vertex		
bus-route problem (variation)	**edge decomposition** into spanning trees		

Further reading

Most of the material in this unit is well covered in the literature. In particular, standard material on planarity and colouring can be found in all the graph theory books mentioned in *Graphs 1*.

Further material on the solution of the four colour problem can be found in the article:

K. Appel and W. Haken, The solution of the four-color-map problem, *Scientific American* **237** No. 4 (October 1977), 108–121,

and in:

T. L. Saaty and P. C. Kainen, *The Four-Color Problem: Assaults and Conquest*, 2nd. edition, Dover, 1986.

(This book also contains a lot of other material on vertex colourings.)

Many of the topics in this unit are discussed in:

G. Chartrand and L. Lesniak, *Graphs & Digraphs*, 3rd. edition, Wadsworth & Brooks/Cole, 1996.

Domination problems are discussed (under the name of *stability problems*) in:

C. Berge, *Graphs*, 2nd. edition, North-Holland, 1985.

References to the refuse collection and other similar problems may be found in:

F. S. Roberts, *Discrete Mathematical Models, with Applications to Social, Biological and Environmental Problems*, Prentice-Hall, 1976.

A full discussion of planarity and its generalizations is given in:

G. Ringel, *Map Color Theorem*, Springer-Verlag, 1974,

and a discussion of planarity algorithms is given in the article:

J. Hopcroft and R. E. Tarjan, Efficient planarity testing, *J. Assoc. Comput. Mach.* **21** (1974), 549–568.

Expository articles by various authors on planarity and its generalizations, the four colour theorem and edge colourings may be found in:

L. W. Beineke and R. J. Wilson (eds.), *Selected Topics in Graph Theory*, Academic Press, 1978,

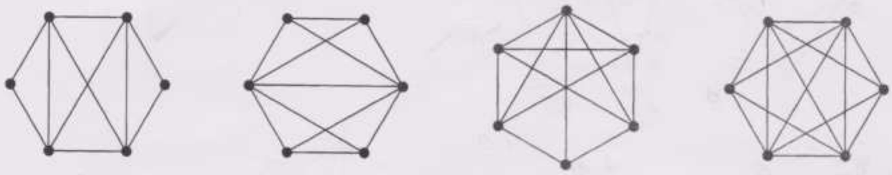
and historical information on planarity and the four colour theorem may be found in:

N. L. Biggs, E. K. Lloyd and R. J. Wilson, *Graph Theory 1736–1936*, paperback edition, Clarendon Press, 1998 (first published 1976).

Exercises

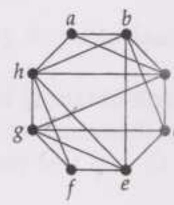
Section 1

1.1 Decide which of the following graphs are planar.



(a) (b) (c) (d)

For each planar graph, give a plane drawing and verify Euler's formula. For each non-planar graph, verify Kuratowski's theorem by finding a subgraph that is a subdivision of K_5 or $K_{3,3}$.



1.2 By finding a plane drawing, show that the graph in the margin is planar. Verify Euler's formula for your plane drawing.

1.3 Let G be a simple connected planar graph with n (≥ 3) vertices and m edges, and let g be the length of the shortest cycle in G.

(a) Prove that

$$m \leq g(n-2)/(g-2).$$

Hint Use the handshaking lemma for planar graphs.

(b) Use the above result to prove that K_5, $K_{3,3}$ and the Petersen graph are non-planar.

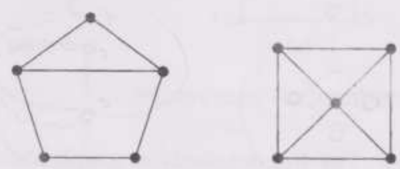
1.4 Prove the following statement.

5 faces If $f \leq 11$

Let G be a simple connected planar graph with n vertices. If $n \leq 11$, then G contains at least one vertex of degree 4 or less.

Hint Use a proof by contradiction.

1.5 Draw the dual of each of the plane drawings of planar graphs shown in the margin.

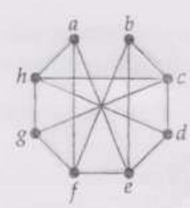


(a) (b)

1.6 Dualize the statement given in Exercise 1.4.

1.7 Let G be a plane drawing of a connected planar graph. Prove that G is bipartite if and only if its dual G^* is Eulerian.

1.8 Use the cycle method to determine whether the graph in the margin is planar.



Hint Find a Hamiltonian cycle and a plane drawing of it.

Section 2

2.1 There was once a king with five sons. In his will he stated that after his death the sons should divide the kingdom into five regions in such a way that the boundary of each region should have a boundary line in common with each of the other four regions. Can the terms of the will be satisfied?

This is the original version of Möbius's problem given in Problem 1.4.

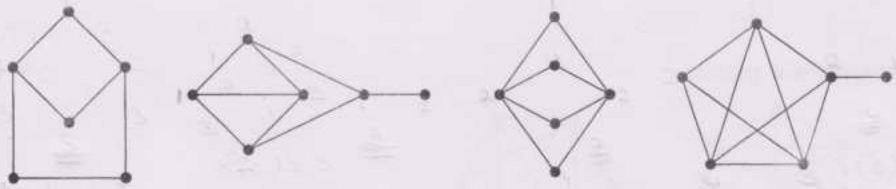
2.2 Let G be a map. Prove that:

(a) if the faces of G can be 2-coloured, then G is an Eulerian graph;

(b) if the faces of G can be 3-coloured and G is 3-regular, then each face of G has even degree.

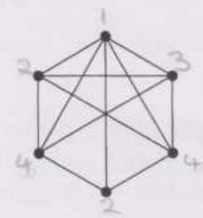
Section 3

3.1 Determine $\chi(G)$ for each of the following graphs G.



(a) (b) (c) (d)

3.2 Consider the graph G in the margin.



(a) Use results from Section 3.1 to obtain lower and upper bounds for $\chi(G)$.

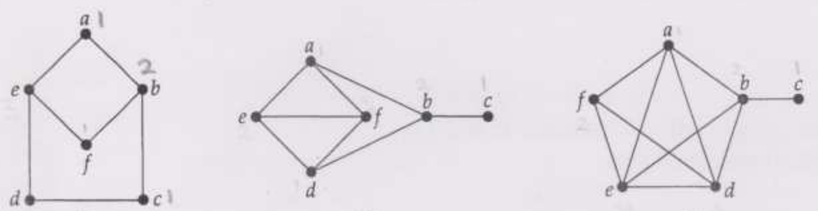
(b) What is the actual value of $\chi(G)$?

3.3 Let G be the graph obtained by removing an edge from the complete graph K_n. By Brooks' theorem, we know that $\chi(G) \le n - 1$. Give a method for $(n - 1)$-colouring G, and test your method by 6-colouring K_7 with one edge removed.

3.4 Prove that if G is an r-regular graph with n vertices, then

$\chi(G) \ge n/(n - r)$.

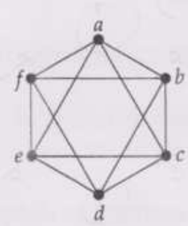
3.5 Use the greedy colouring algorithm to colour the vertices of each of the following labelled graphs.



(a) (b) (c)

Comment on your results.

3.6 For the octahedron graph shown in the margin, find, if possible:



(a) a vertex decomposition in which no two vertices in the same subset are adjacent;

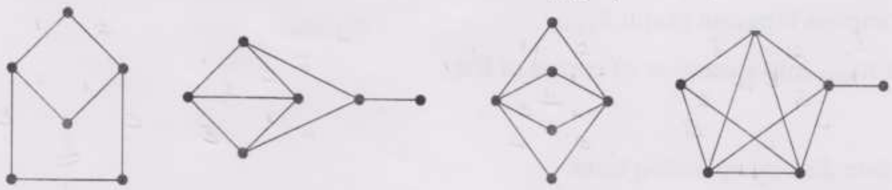
(b) a vertex decomposition in which each subset contains a vertex adjacent to each of the other vertices in the subset.

To which topic in Section 3 does each of these decompositions correspond?

3.7 A youth club organizer wishes to arrange outings to the Zoo for nine children: Andrew, Bill, Catherine, Deirdre, Edward, Fiona, Gina, Harry and Iris. Unfortunately, Catherine refuses to go on an outing with any of the boys, Andrew will not go if there are any girls (except Deirdre), Edward and Harry must not be allowed to go together since they will cause havoc, Fiona cannot stand Bill or Gina, and Bill and Edward both dislike Iris. Express this information in terms of a suitable graph, find the minimum number of outings needed, and write down the corresponding vertex decomposition.

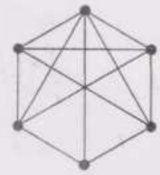
Section 4

4.1 Determine $\chi'(G)$ for each of the following graphs G.



(a) (b) (c) (d)

4.2 Consider the graph G in the margin.



(a) Use Vizing's theorem to obtain lower and upper bounds for $\chi'(G)$.

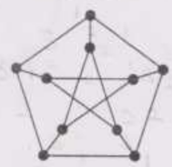
(b) What is the actual value of $\chi'(G)$?

4.3 Consider the graph G in the margin.

(a) Use the theorems of Vizing and Shannon to obtain lower and upper bounds for $\chi'(G)$.

(b) What is the actual value of $\chi'(G)$?

4.4 Prove that the Petersen graph has chromatic index 4.



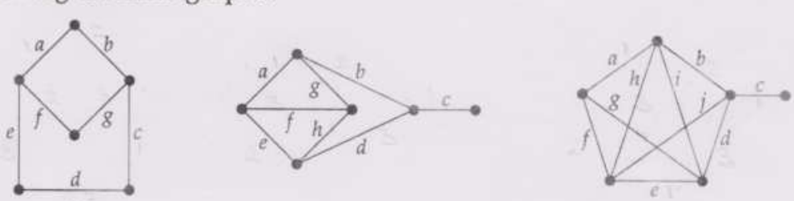
Hint Apply Vizing's theorem. Then assume that the chromatic index is 3, and note that there is essentially only one way to 3-edge colour the outside pentagon.

4.5

(a) Let G be a 3-regular Hamiltonian graph. Show that $\chi'(G) = 3$.

(b) Let G be a 3-regular map. Show that $\chi'(G) = 3$.

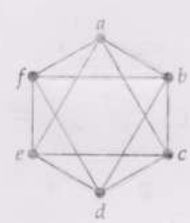
4.6 Use the greedy colouring algorithm to colour the edges of each of the following labelled graphs.



(a) (b) (c)

Comment on your results.

4.7 For the octahedron graph shown in the margin, find, if possible:



(a) an edge decomposition into disjoint cycles;

(b) an edge decomposition into planar subgraphs;

(c) an edge decomposition in which no two edges in any subset meet;

(d) an edge decomposition into connected subgraphs that include every vertex;

(e) an edge decomposition into spanning trees.

To which topic in Section 4 does each of these decompositions correspond?

4.8 Five students are to be examined by four tutors:

tutor A must examine students a, b and e;

tutor B must examine students a, c and d;

tutor C must examine students b, c and e;

tutor D must examine students b, c and d.

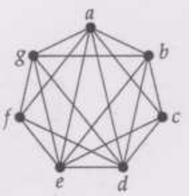
If each examination takes the same amount of time, find the minimum number of examination periods needed, and devise a suitable schedule.

4.9 Show how the complete graph K_7 can be 'printed' in two layers, and write down a corresponding edge decomposition.

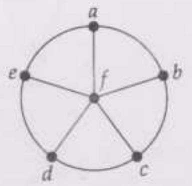
4.10 Determine the thickness of the complete bipartite graph $K_{10,40}$.

Hint To obtain an upper bound, split $K_{10,40}$ into a number of copies of the planar graph $K_{2,40}$.

4.11 Decompose the following graph into disjoint spanning trees.

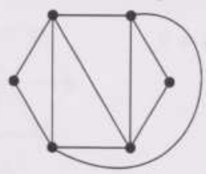


4.12 Verify the statement of Theorem 4.9 for the graph shown below.



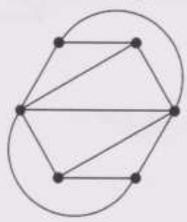
Solutions to the exercises

1.1 Graph (a) is planar: a plane drawing is as follows.



For this graph, $n = 6$, $m = 10$, $f = 6$, and $n - m + f = 6 - 10 + 6 = 2$; this verifies Euler's formula in this case.

Graph (b) is also planar: a plane drawing is as follows.

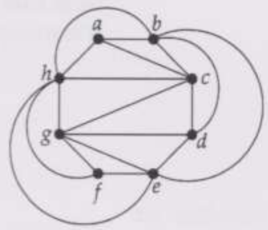


For this graph, $n = 6$, $m = 11$, $f = 7$, and $n - m + f = 6 - 11 + 7 = 2$; this verifies Euler's formula in this case.

Graph (c) is non-planar: it clearly contains $K_{3,3}$, and it also contains a subdivision of K_5, as can be seen by removing the vertical edge through the centre.

Graph (d) is non-planar: it contains $K_{3,3}$ (the two vertices of degree 4 and any vertex of degree 5 form one set, the remaining three vertices of degree 5 form the other), and it also contains K_5, as can be seen by removing one vertex of degree 4 and its incident edges.

1.2 A plane drawing is as follows.



For this graph, $n = 8$, $m = 18$, $f = 12$, and $n - m + f = 8 - 18 + 12 = 2$; this verifies Euler's formula in this case.

1.3

(a) For a plane drawing of G with f faces, it follows from the handshaking lemma for planar graphs that

$$2m \geq gf,$$

since the degree of each face is at least g.

Substituting for f from Euler's formula $f = m - n + 2$, we obtain

$$2m \geq gm - gn + 2g$$

and hence

$$g(n-2) \geq m(g-2),$$

giving

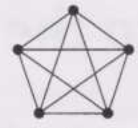
$$m \leq g(n-2)/(g-2), \qquad (*)$$

as required.

(b) If K_5 were planar, then inequality (*) would become (with $m = 10$, $n = 5$ and $g = 3$)

$$10 \le 3(5-2)/(3-2) \quad \text{or} \quad 10 \le 9,$$

which is FALSE. Thus K_5 is non-planar.

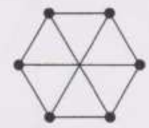


K_5

If $K_{3,3}$ were planar, then inequality (*) would become (with $m = 9$, $n = 6$ and $g = 4$)

$$9 \le 4(6-2)/(4-2) \quad \text{or} \quad 9 \le 8,$$

which is FALSE. Thus $K_{3,3}$ is non-planar.

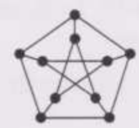


$K_{3,3}$

If the Petersen graph were planar, then inequality (*) would become (with $m = 15$, $n = 10$ and $g = 5$)

$$15 \le 5(10-2)/(5-2) \quad \text{or} \quad 15 \le 40/3,$$

which is FALSE. Thus the Petersen graph is non-planar.



1.4 Suppose that each vertex of G has degree 5 or more. Then, by the handshaking lemma for graphs,

$$2m \ge 5n.$$

But, by Corollary 1.1, we have

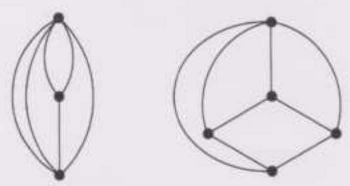
$$m \le 3n - 6.$$

Combining these two inequalities, we obtain

$$5n \le 6n - 12 \quad \text{or} \quad n \ge 12,$$

which is FALSE when $n \le 11$. This contradiction shows that G must have at least one vertex of degree 4 or less.

1.5 The duals are as follows.



(a) (b)

Note that for graph (b) the dual graph is isomorphic to the original graph.

1.6 The dual statement is as follows.

Let G be a connected planar graph and let f be the number of faces in a plane drawing of G. If $f \le 11$ and if G has no vertices of degree 1 or 2, then G contains at least one face of degree 4 or less.

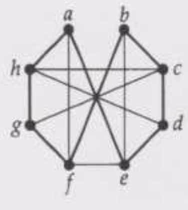
1.7 If G is bipartite, then each cycle of G has even length. By duality, each cutset of G^* has an even number of edges; in particular, each vertex of G^* has even degree. Thus, G^* is Eulerian.

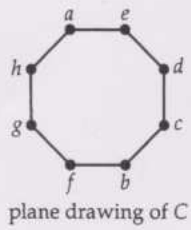
See *Graphs 1*, Problem 1.21.

Conversely, if G^* is Eulerian, then each vertex of G^* has even degree. By duality, each face of G has even degree. It follows that each cycle of G has even length, and so G is bipartite.

See *Graphs 1*, Theorem 2.1.

1.8 Let C be the Hamiltonian cycle *aedcbfgha*; we give a plane drawing of C.





plane drawing of C

af, be
cg, ch
dh, ef

edges not in C

We list the edges which do not belong to C. list: af, be, cg, ch, dh, ef

We put the first edge in the list, af, in a set A. $A = \{af, \ldots\}$

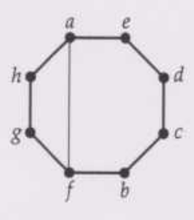
We delete this edge from the list. list: be, cg, ch, dh, ef

The edge af is incompatible with cg, ch and dh, so we put the edges cg, ch and dh in a set B. $B = \{cg, ch, dh, \ldots\}$

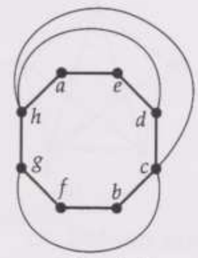
We check and find that all the edges in B are compatible with each other. CHECK ✔

We delete the edges cg, ch and dh from the list. list: be, ef

We now have the following situation.



edges in A



edges in B

We consider the edge cg in B.

The edge cg is incompatible with be and ef, so we put the edges be and ef in A. $A = \{af, be, ef, \ldots\}$

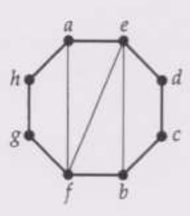
We check and find that all the edges in A are compatible with each other. CHECK ✔

We delete the edges be and ef from the list.

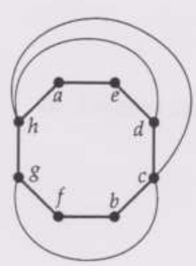
The list is now empty, and we have:

$A = \{af, be, ef\}$;

$B = \{cg, ch, dh\}$.



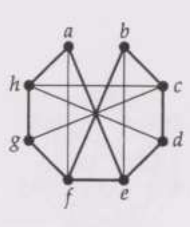
edges in A



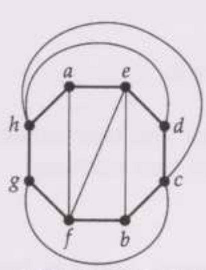
edges in B

All the edges in A are compatible and all the edges in B are compatible, so G is planar.

To obtain a plane drawing of G, we combine the above two figures as follows.



graph G



plane drawing of G

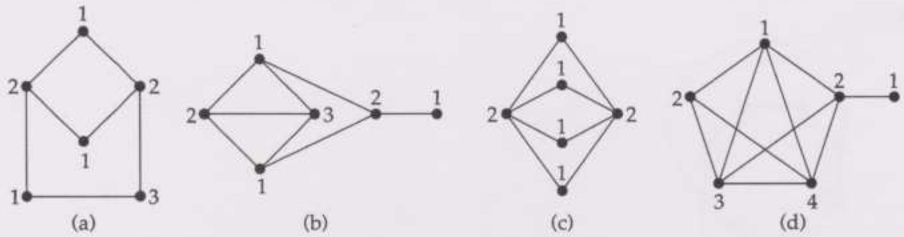
2.1 If it were possible to satisfy the conditions of the will, then it would be possible to find five mutually neighbouring regions in the plane. It would then follow, by duality, that the complete graph K_5 can be drawn in the plane without crossings. Since this is not the case, the terms of the will cannot be satisfied.

2.2

(a) For any vertex v, the number of faces surrounding v must be even, since they can be coloured with two colours. It follows that each vertex has even degree, and so G is Eulerian.

(b) For any face, surrounding faces must alternate in colour, and so the number of such faces must be even. It follows that each face has even degree.

3.1 Possible vertex colourings are given below.



(a) Since the graph contains C_5, at least three colours are needed, so $\chi(G) \geq 3$.

A 3-colouring is shown above, so $\chi(G) \leq 3$.

Thus $\chi(G) = 3$.

(b) Since the graph contains a triangle (K_3), at least three colours are needed, so $\chi(G) \geq 3$.

A 3-colouring is shown above, so $\chi(G) \leq 3$.

Thus $\chi(G) = 3$.

(c) Since the graph contains an edge, at least two colours are needed, so $\chi(G) \geq 2$.

A 2-colouring is shown above, so $\chi(G) \leq 2$.

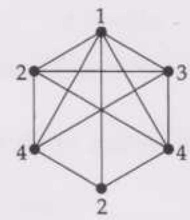
Thus $\chi(G) = 2$.

(d) Since the graph contains K_4, at least four colours are needed, so $\chi(G) \geq 4$.

A 4-colouring is shown above, so $\chi(G) \leq 4$.

Thus $\chi(G) = 4$.

3.2

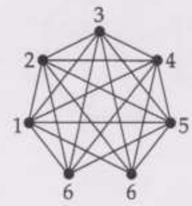


(a) Since the graph contains K_4, at least four colours are needed, so $\chi(G) \geq 4$.

By Brooks' theorem with $d = 5$, $\chi(G) \leq 5$.

Thus $4 \leq \chi(G) \leq 5$.

(b) A 4-colouring of G is shown in the margin, so $\chi(G) = 4$.



3.3 Assign a different colour to each vertex, except for the two vertices that are not adjacent, and assign the final colour to these two vertices.

For $n = 7$, we obtain the colouring shown in the margin.

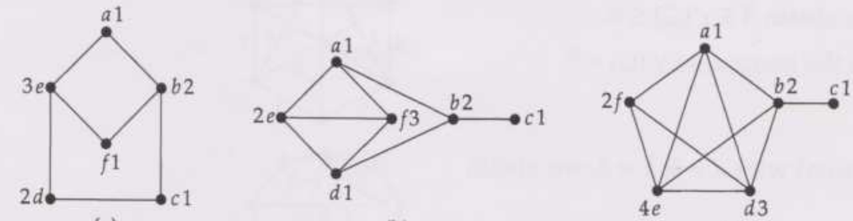
3.4 Suppose that the graph G is r-regular with n vertices, and has been coloured with $\chi(G)$ colours.

Since each vertex must be assigned a different colour from its r neighbours, there must be at most $n - r$ vertices of the same colour. But there are $\chi(G)$ colours used altogether, and so

$$\chi(G) \times (n - r) \geq n.$$

The result follows.

3.5 We obtain the following vertex colourings.

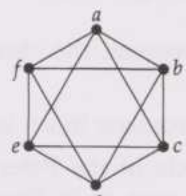


(a) (b) (c)

In each case, we obtain a colouring with $\chi(G)$ colours.

See Exercise 3.1 for the values of $\chi(G)$.

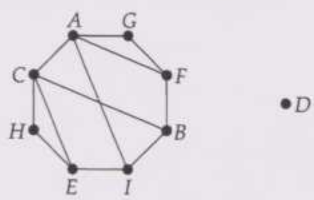
3.6 For each part there are several possible decompositions. We give just one in each case:



(a) $\{a, d\}$, $\{b, e\}$, $\{c, f\}$ — the map colouring and refuse collection problems;

(b) $\{a, b, c\}$, $\{d, e, f\}$ — the domination problem.

3.7 In the following graph, the vertices represent the nine children, and the edges join pairs of children who will not go on the same outing.



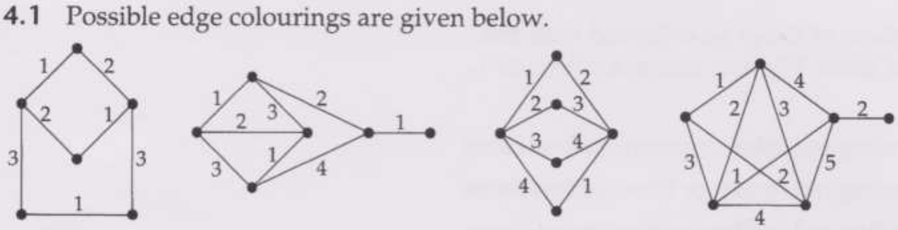
The problem is to find a vertex decomposition of this graph in which no two vertices in the same subset are adjacent — that is, a vertex colouring. Since the graph contains the triangles $AFGA$ and $CEHC$, at least three colours (outings) are needed. One possible arrangement of three outings is as follows.

Outing 1 Andrew, Bill, Edward

Outing 2 Catherine, Fiona, Iris

Outing 3 Deirdre, Gina, Harry

4.1 Possible edge colourings are given below.



(a) (b) (c) (d)

(a) Since the graph contains C_5, at least three colours are needed, so $\chi'(G) \geq 3$.

A 3-edge colouring is shown above, so $\chi'(G) \leq 3$.

Thus $\chi'(G) = 3$.

(b) By Vizing's theorem with $d = 3$, we obtain $3 \leq \chi'(G) \leq 4$. However, G cannot be edge coloured with three colours, so $\chi'(G) = 4$.

(c) Since the graph contains a vertex of degree 4, at least four colours are needed, so $\chi'(G) \geq 4$.

A 4-edge colouring is shown above, so $\chi'(G) \leq 4$.

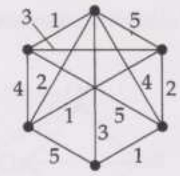
Thus $\chi'(G) = 4$.

Alternatively, the result follows from König's theorem, since G is bipartite.

(d) By Vizing's theorem with $d = 4$, we obtain $4 \leq \chi'(G) \leq 5$. However, G cannot be edge coloured with four colours, so $\chi'(G) = 5$.

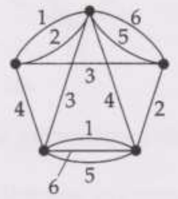
4.2

(a) By Vizing's theorem with $d = 5$, we obtain $5 \le \chi'(G) \le 6$.



(b) A 5-edge colouring of G is shown in the margin, so $\chi'(G) = 5$.

4.3



(a) By Vizing's theorem (extended version) with $d = 6$, $h = 3$, we obtain $6 \le \chi'(G) \le 9$.

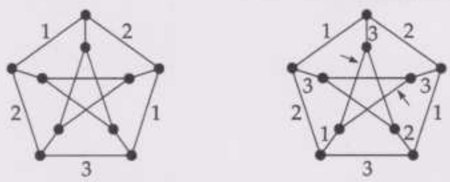
By Shannon's theorem with $d = 6$, we obtain $6 \le \chi'(G) \le 9$.

(b) A 6-edge colouring of G is shown in the margin, so $\chi'(G) = 6$.

4.4 By Vizing's theorem, the chromatic index of the Petersen graph is either 3 or 4.

Suppose that the chromatic index is 3.

The bounding 5-cycle of the Petersen graph can be 3-edge coloured in essentially only one way, as in diagram (a) below. The 'spokes' joining the outer and inner cycles must then be coloured as in diagram (b).



(a) (b)

But it is now impossible to colour both the arrowed edges with colour 2, since they meet at a vertex. Thus it is impossible to 3-edge colour the Petersen graph, and so the chromatic index is 4.

4.5

(a) Since G is 3-regular, at least three colours are needed, so $\chi'(G) \ge 3$.

To obtain a 3-edge colouring of G, we alternately colour the edges of a Hamiltonian cycle using colours 1 and 2, and then colour the remaining edges with colour 3.

Note that G has an even number of vertices, by the handshaking lemma, and thus any Hamiltonian cycle has an even number of edges.

(b) By the four colour theorem, the faces of G can be coloured with the four colours red, blue, yellow and green. We now assign a colour to each edge:

assign colour 1 to each edge bounding red/blue or green/yellow faces;

assign colour 2 to each edge bounding red/green or blue/yellow faces;

assign colour 3 to each edge bounding red/yellow or blue/green faces.

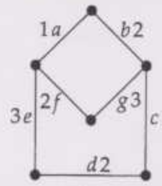
This gives a 3-edge colouring of G.

Since G is 3-regular, $\chi'(G) \ge 3$.

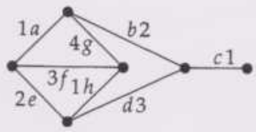
Thus $\chi'(G) = 3$.

For example, the 4-colouring of the faces of the following map gives rise to the 3-edge colouring shown below.

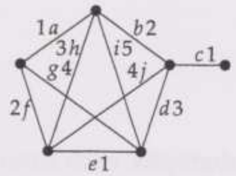
4.6 We obtain the following edge colourings.



(a)



(b)



(c)

In each case, we obtain an edge colouring with $\chi'(G)$ colours.

See Exercise 4.1 for the values of $\chi'(G)$.

4.7 For each part there are several decompositions. We give just one in each case:

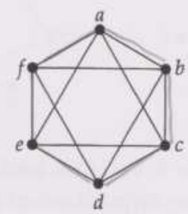
(a) $\{ab, af, bc, cd, de, ef\}$, $\{ac, ae, ce\}$, $\{bd, bf, df\}$ — decomposition of an Eulerian graph into cycles;

(b) $\{ab, ac, ae, af, bc, bd, bf, cd, ce, de, df, ef\}$ — the printed circuits problem; every edge is included, because the graph is planar;

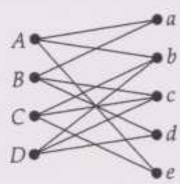
(c) $\{ab, cd, ef\}$, $\{ac, bf, de\}$, $\{ae, bc, df\}$, $\{af, bd, ce\}$, — the examination scheduling problem;

(d) $\{ab, bc, cd, de, ef\}$, $\{ac, ae, af, bd, bf, ce, df\}$ — the first bus route problem;

(e) there is no solution because 12 (the number of edges) is not a multiple of 5 (the number of edges in a spanning tree) — the second bus route problem.



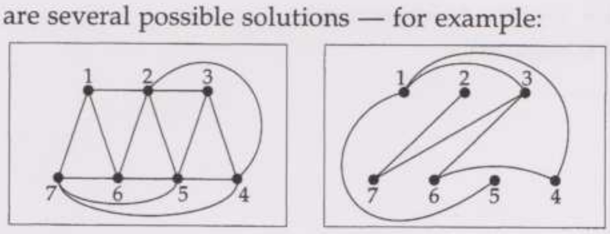
4.8 The corresponding bipartite graph is given below.



Since this graph is bipartite with maximum vertex degree 3, three examination periods are needed, by König's theorem. A schedule that involves exactly three examination periods can be obtained by decomposing the edges of this graph into three subsets in each of which the edges do not meet. One such schedule is as follows.

tutor	A	B	C	D
first examination period	a	c	b	d
second examination period	b	a	e	c
third examination period	e	d	c	b

4.9 There are several possible solutions — for example:



The corresponding edge decomposition is:

$\{12, 16, 17, 23, 24, 25, 26, 34, 35, 45, 47, 56, 57, 67\}$, $\{13, 14, 15, 27, 36, 37, 46\}$.

4.10 By Theorem 4.8, part (b),

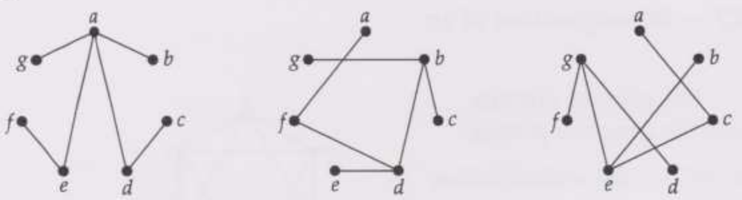
$$t(K_{10,40}) \geq \lceil (10 \times 40)/(20 + 80 - 4) \rceil$$
$$= \lceil 400/96 \rceil = \lceil 4.16... \rceil = 5,$$

and so $t(K_{10,40}) \geq 5$.

But we can split $K_{10,40}$ into five planar subgraphs, each isomorphic to $K_{2,40}$. Thus $t(K_{10,40}) \leq 5$.

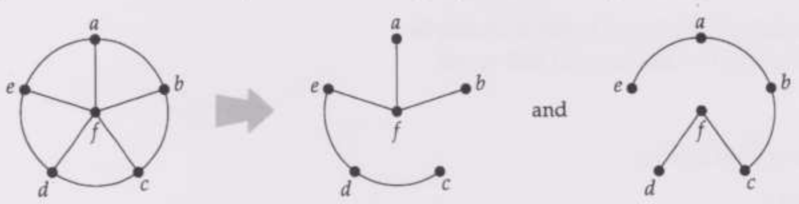
Combining these two inequalities, we obtain $t(K_{10,40}) = 5$.

4.11 The graph can be decomposed into disjoint spanning trees, as follows.



Several other decompositions are possible.

4.12 The graph has 6 vertices and 10 edges, and so $s = 2$; the following diagram shows a decomposition of the graph into two spanning trees.



We now check the condition in the statement of Theorem 4.9.

In order to disconnect the graph into:

2 components, we must remove at least 3 edges

— for example, *ab*, *ae* and *af*;

3 components, we must remove at least 5 edges

— for example, *ab*, *ae*, *af*, *bc* and *bf*;

4 components, we must remove at least 7 edges

— for example, *ab*, *ae*, *af*, *bc*, *bf*, *cd* and *cf*;

5 components, we must remove at least 9 edges

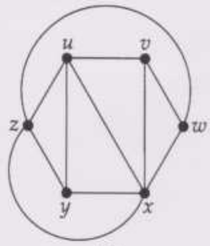
— for example, *ab*, *ae*, *af*, *bc*, *bf*, *cd*, *cf*, *de* and *df*;

6 components, we must remove all 10 edges.

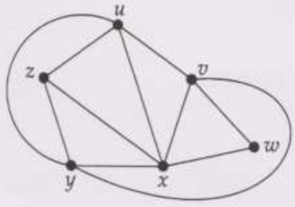
Thus, for each positive integer $k \leq 6$, at least $(k-1) \times s = 2(k-1)$ edges must be removed in order to disconnect the graph into k components.

Solutions to the problems

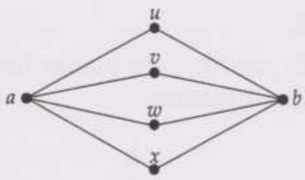
Solution 1.1



(a)



(b)

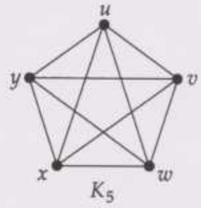


(c)

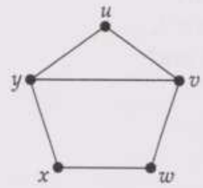
Solution 1.2

The nine connections are not possible because the corresponding graph is $K_{3,3}$, which is non-planar.

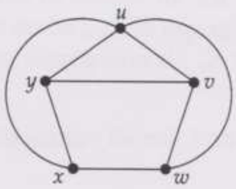
Solution 1.3



(a)



(b)



(c)

In any plane drawing of K_5, the cycle $uvwxyu$ in diagram (a) must appear as a pentagon. The edge vy must lie either inside or outside the pentagon. Since the argument is similar in each case, we assume that vy lies inside the pentagon, as in diagram (b).

Since the edges ux and uw cannot cross vy, they must lie *outside* the pentagon, as in diagram (c). But the edge vx cannot cross uw, and the edge wy cannot cross ux, so both vx and wy must lie *inside* the pentagon, and must therefore cross. Since this is not allowed, we deduce that K_5 has no plane drawing — that is, K_5 is non-planar.

Solution 1.4

No. The corresponding graph is K_5, which is non-planar.

Solution 1.5

(a) This statement is TRUE, since if G is a planar graph, then we can draw G in the plane without crossings. If we now remove the vertices and edges not included in the subgraph, then we obtain a plane drawing of the subgraph.

(b) This statement is FALSE; for example, the graph $K_{3,3}$ is non-planar, whereas the cycle graph C_6, a subgraph of $K_{3,3}$, is planar.

(c) This statement is FALSE; for example, the graph C_6 is planar, whereas the graph $K_{3,3}$, which contains C_6, is non-planar.

(d) This statement is TRUE, since if G is a planar graph, then G cannot have a non-planar subgraph, by part (a).

Solution 1.6

(a) All trees are planar.

All trees are plane trees!

(b) All cycle graphs are planar.

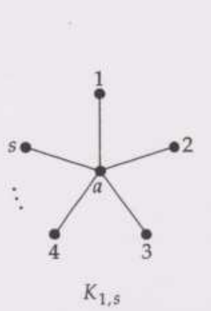
(c) Using the results of Problem 1.5, parts (a) and (d), we deduce that:

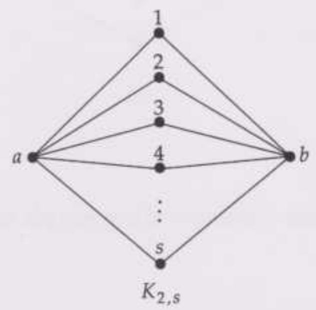
since K_4 is planar and K_n is a subgraph of K_4 when $n \leq 4$, K_n is planar when $n \leq 4$;

since K_5 is non-planar, and K_5 is a subgraph of K_n when $n \geq 5$, K_n is non-planar when $n \geq 5$.

Thus K_n is planar for $n = 1, 2, 3, 4$.

(d) The complete bipartite graphs $K_{1,s}$ and $K_{2,s}$ are planar for all values of s, as illustrated in the following diagrams.

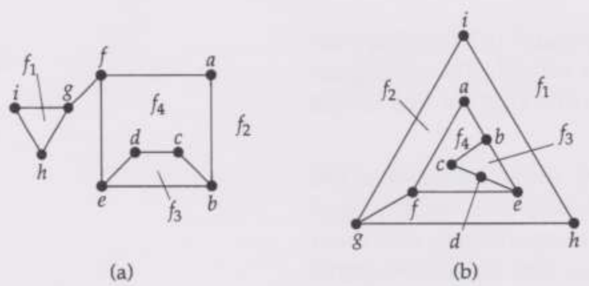




(e) The graphs $K_{1,s}$ and $K_{2,s}$ are planar for all values of s, by part (d). Since $K_{3,3}$ is non-planar and is a subgraph of $K_{r,s}$ when $r \geq 3$ and $s \geq 3$, it follows from Problem 1.5(d) that $K_{r,s}$ is non-planar when $r \geq 3$ and $s \geq 3$.

Thus $K_{r,s}$ is planar only when $r = 1$ or 2, for all values of s.

Solution 1.7



Solution 1.8

(a) There are 10 edges and six faces with face degrees 1, 2, 3, 3, 4, 7, and

$$1 + 2 + 3 + 3 + 4 + 7 = 20 = 2 \times 10.$$

(b) There are 11 edges and seven faces with face degrees 3, 3, 3, 3, 3, 3, 4, and

$$3 + 3 + 3 + 3 + 3 + 3 + 4 = 22 = 2 \times 11.$$

(c) There are 10 edges and six faces with face degrees 3, 3, 3, 3, 4, 4, and

$$3 + 3 + 3 + 3 + 4 + 4 = 20 = 2 \times 10.$$

Solution 1.9

(a) There are 6 vertices, 10 edges and 6 faces, and so the required value is

$$6 - 10 + 6 = 2.$$

(b) There are 6 vertices, 11 edges and 7 faces, and so the required value is

$$6 - 11 + 7 = 2.$$

(c) There are 6 vertices, 10 edges and 6 faces, and so the required value is

$$6 - 10 + 6 = 2.$$

Solution 1.10

(a) There are 6 vertices, 12 edges and 8 faces, and

$$n - m + f = 6 - 12 + 8 = 2.$$

(b) There are $k + 1$ vertices, $2k$ edges and $k + 1$ faces, and

$$n - m + f = (k + 1) - 2k + (k + 1) = 2.$$

(c) There are $k + 2$ vertices, $2k$ edges and k faces, and

$$n - m + f = (k + 2) - 2k + k = 2.$$

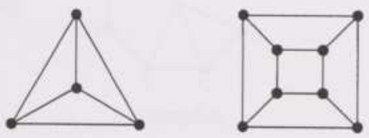
(d) There are $(k + 1)^2$ vertices, $2k(k + 1)$ edges and $k^2 + 1$ faces, and

$$n - m + f = (k + 1)^2 - 2k(k + 1) + (k^2 + 1) = 2.$$

Solution 1.11

In Corollary 1.1, equality occurs when $m = 3n - 6$ or $3n = m + 6$; substituting for n in Euler's formula $n - m + f = 2$, we obtain $2m = 3f$. So equality occurs when G is face-regular of degree 3; for example, K_4 ($m = 6, f = 4$).

In Corollary 1.2, equality occurs when $m = 2n - 4$ or $2n = m + 4$; substituting for n in Euler's formula $n - m + f = 2$, we obtain $2m = 4f$. So equality occurs when G is face-regular of degree 4; for example, the 3-cube ($m = 12, f = 6$).



K_4 3-cube

Solution 1.12

(a) For a plane drawing of G with f faces, it follows from handshaking lemma for planar graphs that

$$2m \geq 5f,$$

since the degree of each face is at least 5.

Substituting for f from Euler's formula $f = m - n + 2$, we obtain

$$2m \geq 5m - 5n + 10 \quad \text{or} \quad 3m \leq 5(n - 2),$$

and hence

$$m \leq \tfrac{5}{3}(n - 2),$$

as required.

(b) If the Petersen graph were planar, then the inequality in part (a) would become (with $m = 15$ and $n = 10$):

$$15 \leq \tfrac{5}{3} \times (10 - 2) = 13\tfrac{1}{3},$$

which is FALSE. Thus the Petersen graph is non-planar.

Solution 1.13

Since G is a simple graph, we can apply Corollary 1.1 to deduce that, if G has n vertices and m edges, then

$$m \leq 3n - 6.$$

Suppose that each vertex of G has degree 6 or more. Then, by the handshaking lemma for graphs,

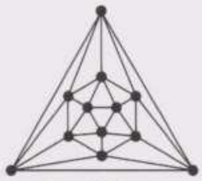
$$2m \geq 6n \quad \text{or} \quad 3n \leq m.$$

Combining these two inequalities, we obtain

$$3n \leq 3n - 6,$$

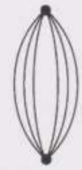
which is FALSE. This contradiction shows that G must have at least one vertex of degree 5 or less.

Solution 1.14



icosahedron

(a)



(b)

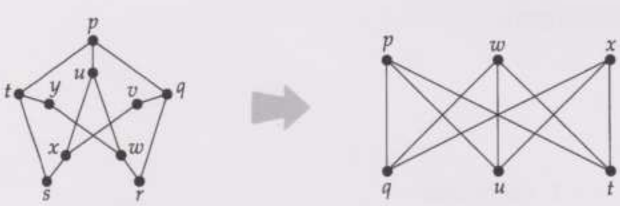
Solution 1.15

(a) Deletion of the edge sw gives the following subgraph.

subgraph $\qquad$ $K_{3,3}$

This is a subdivision of $K_{3,3}$. It follows from Kuratowski's theorem that the given graph is non-planar.

(b) Deletion of the two 'horizontal' edges gives the following subgraph.

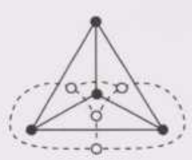


subgraph $\qquad$ $K_{3,3}$

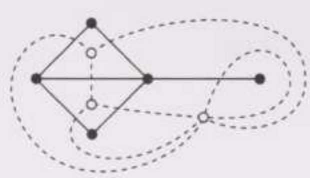
This is a subdivision of $K_{3,3}$. It follows from Kuratowski's theorem that the Petersen graph is non-planar.

Note that the Petersen graph does not contain a subdivision of K_5.

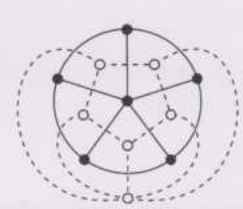
Solution 1.16



(a)



(b)

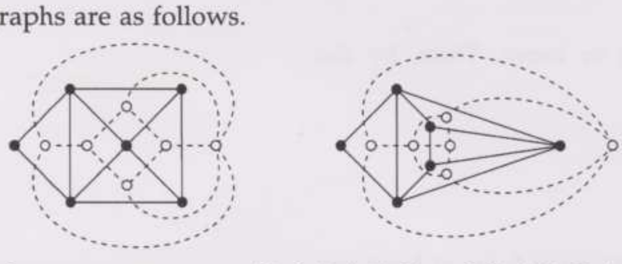


(c)

Notice that in parts (a) and (c) the dual graph is isomorphic to the original graph.

Solution 1.17

The dual graphs are as follows.



Since their degree sequences are (3, 3, 3, 3, 3, 5) and (3, 3, 3, 3, 4, 4), they are not isomorphic.

Solution 1.18

Since a triangle in G corresponds to a cutset with 3 edges in G^*, the dual statement is as follows.

> Let G^* be a connected planar graph with f faces and m edges, and with no cutsets with 1, 2 or 3 edges. Then $m \le 2f - 4$.

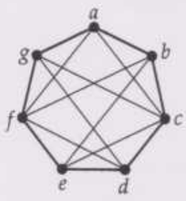
Solution 1.19

(a) We choose C to be the cycle *abcdefga*.

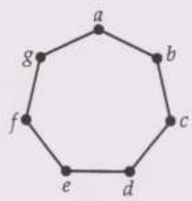
We list the edges which do not belong to C.

list: *ac, af, be, bf, ce, cg, df, dg*

$A = \{ac, \dots\}$



graph G



cycle C

ac	*af*
be	*bf*
ce	*cg*
df	*dg*

edges which do not belong to C

We put the first edge in the list, *ac*, in a set A.

We delete this edge from the list. — list: *af, be, bf, ce, cg, df, dg*

The edge *ac* is incompatible with *be* and *bf*, so we put the edges *be* and *bf* in a set B. — $B = \{be, bf, \dots\}$

We check and find that the edges *be* and *bf* are compatible with each other. — CHECK ✓

We delete the edges *be* and *bf* from the list. — list: *af, ce, cg, df, dg*

We consider the edges *be* and *bf* in B.

The edge *be* is incompatible with *cg*, *df* and *dg*, and the edge *bf* is incompatible with *cg* and *dg*, so we put the edges *cg*, *df* and *dg* in A. — $A = \{ac, cg, df, dg, \dots\}$

We check and find that the edges in A are compatible with each other. — CHECK ✓

We delete the edges *cg*, *df* and *dg* from the list. — list: *af, ce*

The edge *dg* in A is incompatible with *af* and *ce*, so we put the edges *af* and *ce* in B. — $B = \{be, bf, af, ce, \dots\}$

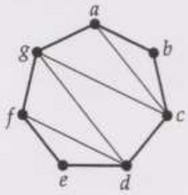
We check and find that all the edges in B are compatible with each other. — CHECK ✓

We delete the edges *af* and *ce* from the list.

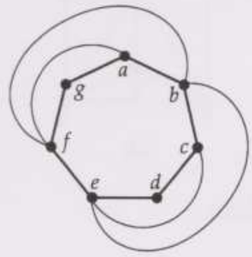
The list is now empty, and we have:

$A = \{ac, cg, df, dg\}$;

$B = \{be, bf, af, ce\}$.



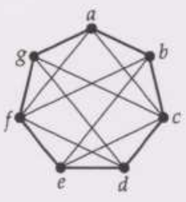
edges in A



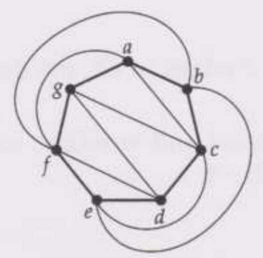
edges in B

All the edges in A are compatible and all the edges in B are compatible, so G is planar.

To obtain a plane drawing of G, we combine the above two figures as follows.

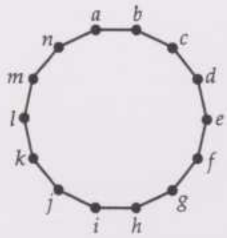


graph G

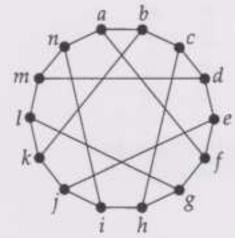


plane drawing of G

(b) The first step is to choose a suitable cycle C in G. In this example, it is natural to choose the cycle *abcdefghijklmna*.



cycle C



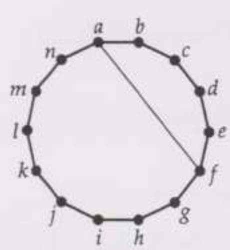
graph G

We list the edges which do not belong to C. list: *af, bk, ch, dm, ej, gl, in*

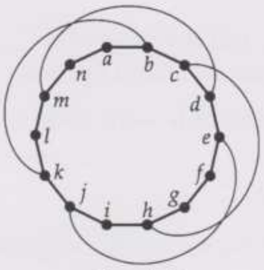
We put the first edge in the list, *af*, in a set A. $A = \{af, \ldots\}$

We delete this edge from the list. list: *bk, ch, dm, ej, gl, in*

The edge *af* in A is incompatible with *bk, ch, dm* and *ej*, so we put the edges *bk, ch, dm* and *ej* into a set B. $B = \{bk, ch, dm, ej, \ldots\}$



edges in A

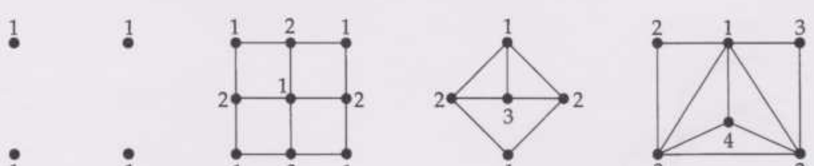


edges in B

We check the compatibility of the edges in B with each other, but find that *bk* and *dm* are incompatible, so G is non-planar. CHECK ✖

Solution 3.1

Possible vertex colourings are given below.



(a) (b) (c) (d)

(a) The vertices can all be coloured with the same colour.

Thus $\chi(G) = 1$.

(b) Since the graph contains an edge, at least two colours are needed, so $\chi(G) \geq 2$.

A 2-colouring is shown above, so $\chi(G) \leq 2$.

Thus $\chi(G) = 2$.

(c) Since the graph contains a triangle (K_3), at least three colours are needed, so $\chi(G) \geq 3$.

A 3-colouring is shown above, so $\chi(G) \leq 3$.

Thus $\chi(G) = 3$.

(d) Since the graph contains K_4, at least four colours are needed, so $\chi(G) \geq 4$.

A 4-colouring is shown above, so $\chi(G) \leq 4$.

Thus $\chi(G) = 4$.

Solution 3.2

(a) The graphs with $\chi(G) = 1$ are those with no edges — that is, the null graphs N_n.

(b) The graphs with $\chi(G) = 2$ are the bipartite graphs (other than N_n), since we can colour their vertices black and white in such a way that each edge joins a black vertex to a white vertex.

Solution 3.3

(a) n;

(b) 2;

(c) 2, if n is even;
3, if n is odd;

(d) 1, if the tree has only one vertex;
2, if the tree has at least two vertices.

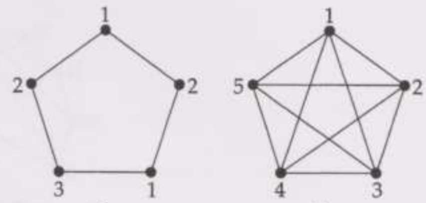
Solution 3.4

(a) This statement is TRUE, because if G contains K_r as a subgraph, then G must contain r mutually adjacent vertices, and these require r colours. So $\chi(G) \geq r$.

(b) This statement is FALSE; for example, the cycle graph C_5 has chromatic number 3, but does not contain a triangle (K_3).

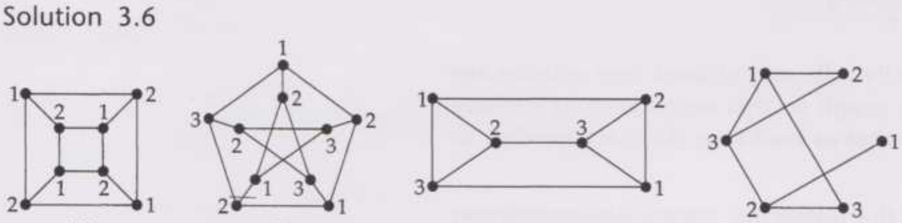
Solution 3.5

The only such graphs G are the cycle graph C_5, with $d = 2$ and $\chi(G) = 3$, and the complete graph K_5, with $d = 4$ and $\chi(G) = 5$.



C_5 K_5

Solution 3.6



(a) (b) (c) (d)

For each graph above:

(a) lower bound: $\chi(G) \geq 2$; upper bound: $\chi(G) \leq 3$; actual value: $\chi(G) = 2$;

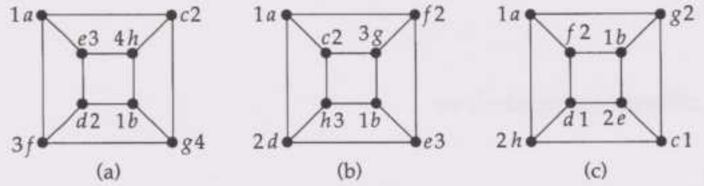
(b) lower bound: $\chi(G) \geq 2$; upper bound: $\chi(G) \leq 3$; actual value: $\chi(G) = 3$;

(c) lower bound: $\chi(G) \geq 3$; upper bound: $\chi(G) \leq 3$; actual value: $\chi(G) = 3$;

(d) lower bound: $\chi(G) \geq 3$; upper bound: $\chi(G) \leq 3$; actual value: $\chi(G) = 3$.

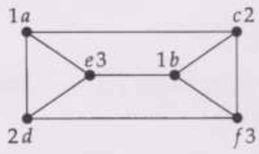
Solution 3.7

We obtain the following vertex colourings with 4, 3 and 2 colours.



(a) (b) (c)

We showed that $\chi(G) = 2$ in Solution 3.6(a), so only colouring (c) uses $\chi(G)$ colours.

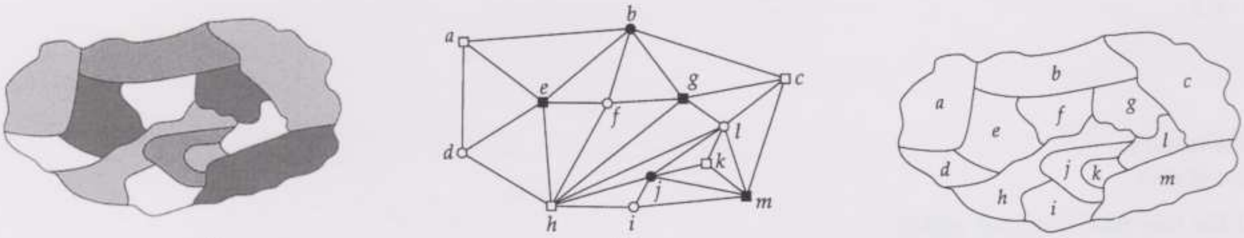


Solution 3.8

We showed that $\chi(G) = 3$ in Solution 3.6(c). A suitable labelling is shown in the margin.

Solution 3.9

(a) There are many possibilities — for example, the colouring shown on the left below.



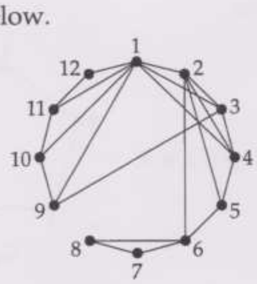
(a) (b)

(b) The corresponding graph is shown above. The colouring in part (a) leads to a vertex decomposition of the required type:

$\{a, c, h, k\}, \{b, j\}, \{d, f, i, l\}, \{e, g, m\}$.

Solution 3.10

The tour graph is given below.



Since vertices 1, 2, 3 and 4 are mutually adjacent, at least four colours are needed to colour the vertices of this graph so that neighbouring vertices are coloured differently. This means that at least four days are needed to schedule the various tours.

In fact, four days are sufficient, as the following vertex decomposition shows:

Monday	tours 1, 5 and 7;
Tuesday	tours 2, 9 and 12;
Wednesday	tours 3, 6 and 11;
Thursday	tours 4, 8 and 10.

Other vertex decompositions are possible.

Solution 3.11

Other minimum dominating sets are

$\{B, D\}, \{B, E\}, \{B, F\}, \{C, F\}, \{D, F\}$.

Solution 3.12

There are several possibilities — for example:

(a) $\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, e\}$;

(b) $\{a, g\}, \{b, h\}, \{c, e\}, \{d, f\}$.

Solution 3.13

(a) 2;

(b) 2.

Solution 3.14

There are several possibilities — for example:

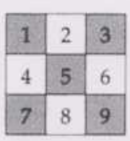
(a) the minimum dominating set $\{a, c\}$ gives rise to the vertex decomposition

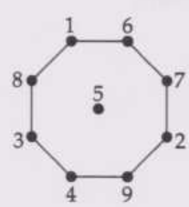
$\{a, b, e\}, \{c, d\}$;

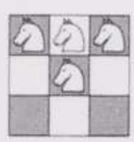
(b) the minimum dominating set $\{a, g\}$ gives rise to the vertex decomposition

$\{a, b, d, e\}, \{g, f, h, c\}$.

Solution 3.15







There are no minimum dominating sets with three vertices.

A minimum dominating set with four vertices is $\{1, 2, 3, 5\}$, giving rise to the vertex decomposition

$\{1, 6, 8\}, \{2, 7, 9\}, \{3, 4\}, \{5\}$.

Other solutions are possible.

Hence the smallest number of knights is 4.

Solution 3.16

(a) 4;

(b) 2.

Solution 3.17

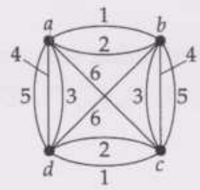
(a) We have dom $(G) = 2$, ind $(G) = 4$, $\chi(G) = 2$ and $n = 8$, so the inequalities state that $2 \leq 4$ and $2 \times 4 \geq 8$, which are correct.

(b) We have dom $(G) = 2$, ind $(G) = 4$, $\chi(G) = 2$ and $n = 7$, so the inequalities state that $2 \leq 4$ and $2 \times 4 \geq 7$, which are correct.

(c) We have dom $(G) = 2$, ind $(G) = 2$, $\chi(G) = 3$ and $n = 6$, so the inequalities state that $2 \leq 2$ and $3 \times 2 \geq 6$, which are correct.

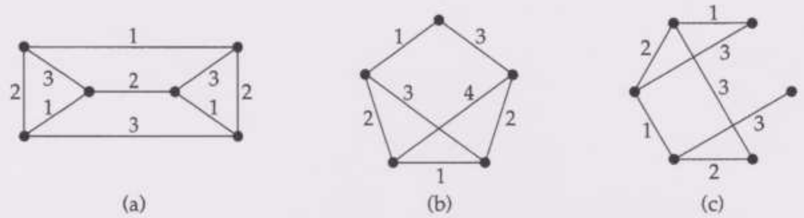
Solution 4.1

Since the degree of each vertex is 6, at least six colours are necessary to colour all the edges of the graph. In fact, six colours are sufficient, as the diagram in the margin shows.



Solution 4.2

Possible edge colourings are given below.



(a) Since the graph contains a vertex of degree 3, at least three colours are needed, so $\chi'(G) \geq 3$.

A 3-edge colouring is shown above, so $\chi'(G) \leq 3$.

Thus $\chi'(G) = 3$.

(b) Since the graph contains a vertex of degree 3, at least three colours are needed, so $\chi'(G) \geq 3$.

However, in this case there is no 3-edge colouring, because three colours are needed to colour the pentagon and a further colour is needed to colour one of the inside edges, so $\chi'(G) \geq 4$.

A 4-edge colouring is shown above, so $\chi'(G) \leq 4$.

Thus $\chi'(G) = 4$.

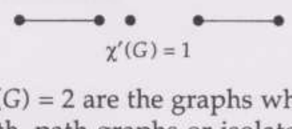
(c) Since the graph contains a vertex of degree 3, at least three colours are needed, so $\chi'(G) \geq 3$.

A 3-edge colouring is shown above, so $\chi'(G) \leq 3$.

Thus $\chi'(G) = 3$.

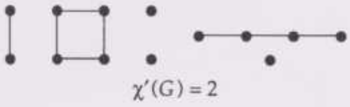
Solution 4.3

(a) The graphs with $\chi'(G) = 1$ are the graphs containing one component which is a single edge and in which each other component is either a single edge or an isolated vertex.



$\chi'(G) = 1$

(b) The graphs with $\chi'(G) = 2$ are the graphs whose components are cycles of even length, path graphs or isolated vertices; at least one component must be a cycle or path graph.



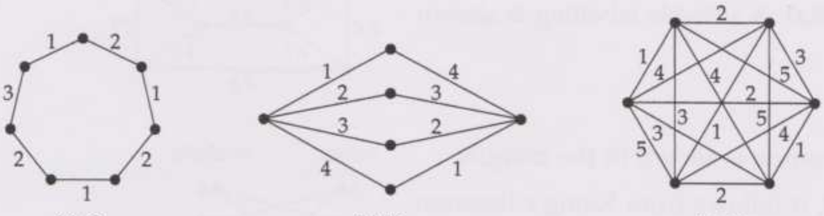
$\chi'(G) = 2$

Solution 4.4

(a) This statement is TRUE, because if G contains a vertex of degree r, then G must contain r edges all of which must be coloured differently, and these require r colours. So $\chi'(G) \geq r$.

(b) This statement is FALSE; for example, the cycle graph C_5 has chromatic index 3, but does not contain a vertex of degree 3.

Solution 4.5



(a) C_7 (b) $K_{2,4}$ (c) K_6

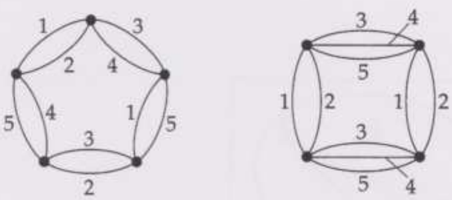
For each graph above:

(a) lower bound: $\chi'(G) \geq 2$; upper bound: $\chi'(G) \leq 3$; actual value: $\chi'(G) = 3$;

(b) lower bound: $\chi'(G) \geq 4$; upper bound: $\chi'(G) \leq 5$; actual value: $\chi'(G) = 4$;

(c) lower bound: $\chi'(G) \geq 5$; upper bound: $\chi'(G) \leq 6$; actual value: $\chi'(G) = 5$.

Solution 4.6



(a) (b)

(a) Vizing's theorem: lower bound: $\chi'(G) \geq 4$; upper bound: $\chi'(G) \leq 6$;
Shannon's theorem: lower bound: $\chi'(G) \geq 4$; upper bound: $\chi'(G) \leq 6$;
actual value: $\chi'(G) = 5$.

(b) Vizing's theorem: lower bound: $\chi'(G) \geq 5$; upper bound: $\chi'(G) \leq 8$;
Shannon's theorem: lower bound: $\chi'(G) \geq 5$; upper bound: $\chi'(G) \leq 7$;
actual value: $\chi'(G) = 5$.

Solution 4.7

In each part, we represent the competition by a complete graph K_n, and the solution is then given by $\chi'(K_n)$.

(a) If $n = 31$, we have $\chi'(K_n) = 31$, by Theorem 4.4.

(b) If $n = 32$, we have $\chi'(K_n) = 31$, by Theorem 4.4.

Thus, in each case, 31 matches are necessary.

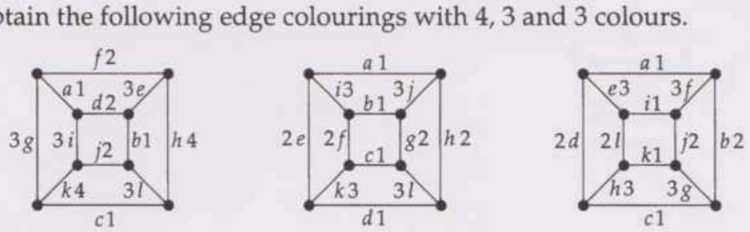
Solution 4.8

In each case, the chromatic index is the maximum vertex degree:

(a) s; (b) 3; (c) k.

Solution 4.9

We obtain the following edge colourings with 4, 3 and 3 colours.

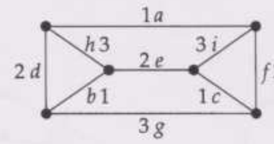


(a) (b) (c)

The actual value of $\chi'(G)$ is 3.

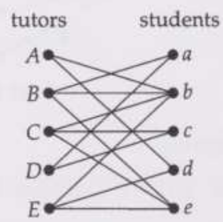
Solution 4.10

We showed that $\chi'(G) = 3$ in Solution 4.2(a). A suitable labelling is shown in the margin.



Solution 4.11

The bipartite graph representing this situation is shown in the margin.

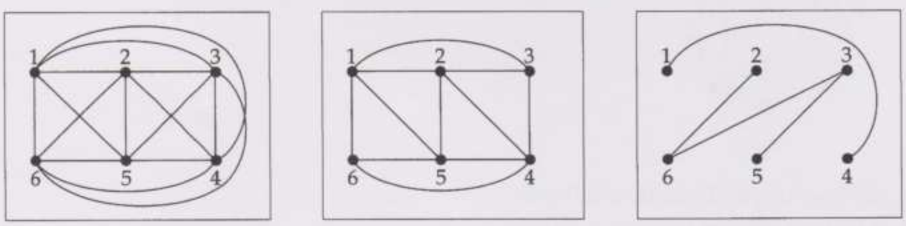


Since this graph has maximum degree 4, it follows from König's theorem that four examination periods are needed. One such schedule is as follows.

tutor	A	B	C	D	E
first examination period	b	a	c	–	e
second examination period	d	b	e	a	–
third examination period	–	e	b	c	d
fourth examination period	–	–	–	–	b

Solution 4.12

The graph K_6 can be 'printed' in two layers, as follows.



K_6 layer 1 layer 2

The corresponding edge decomposition is:

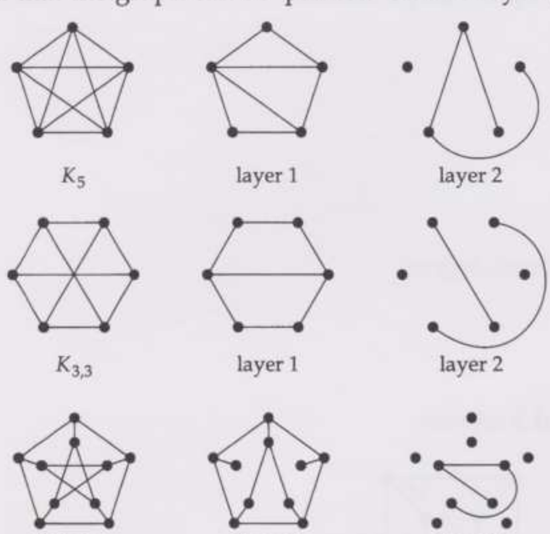
{12, 13, 15, 16, 23, 24, 25, 34, 45, 46, 56}, {14, 26, 35, 36}.

Other solutions are possible.

Solution 4.13

Since each graph is non-planar, the thickness cannot equal 1. But, for each graph, we find that the graph can be 'printed' on two layers, as follows.

Other solutions are possible.



K_5 layer 1 layer 2

$K_{3,3}$ layer 1 layer 2

Petersen layer 1 layer 2

Thus, in each case, the thickness is 2.

Solution 4.14

We use the results stated in the margin at the end of the proofs of parts (a) and (b) of Theorem 4.8.

(a) It follows from the margin note that

$$t(K_{20}) = \lfloor (20+7)/6 \rfloor = \lfloor 27/6 \rfloor = 4.$$

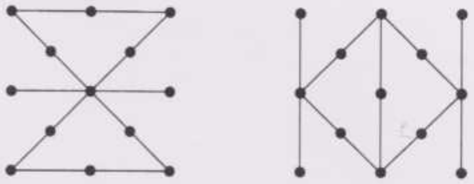
(b) Since $K_{20,20}$ has fewer than 48 vertices, it follows from the margin note that

$$t(K_{20,20}) = \lceil (20 \times 20)/(40+40-4) \rceil = \lceil 400/76 \rceil = \lceil 5.263... \rceil = 6.$$

Solution 4.15

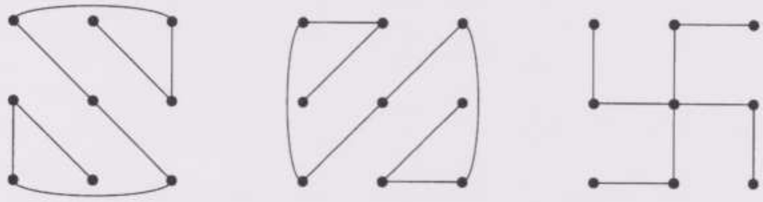
The network has 13 towns and 28 roads, and so $s(G) \le 28/12$; it follows that $s(G) = 1$ or 2.

The following diagram shows that $s(G) = 2$.



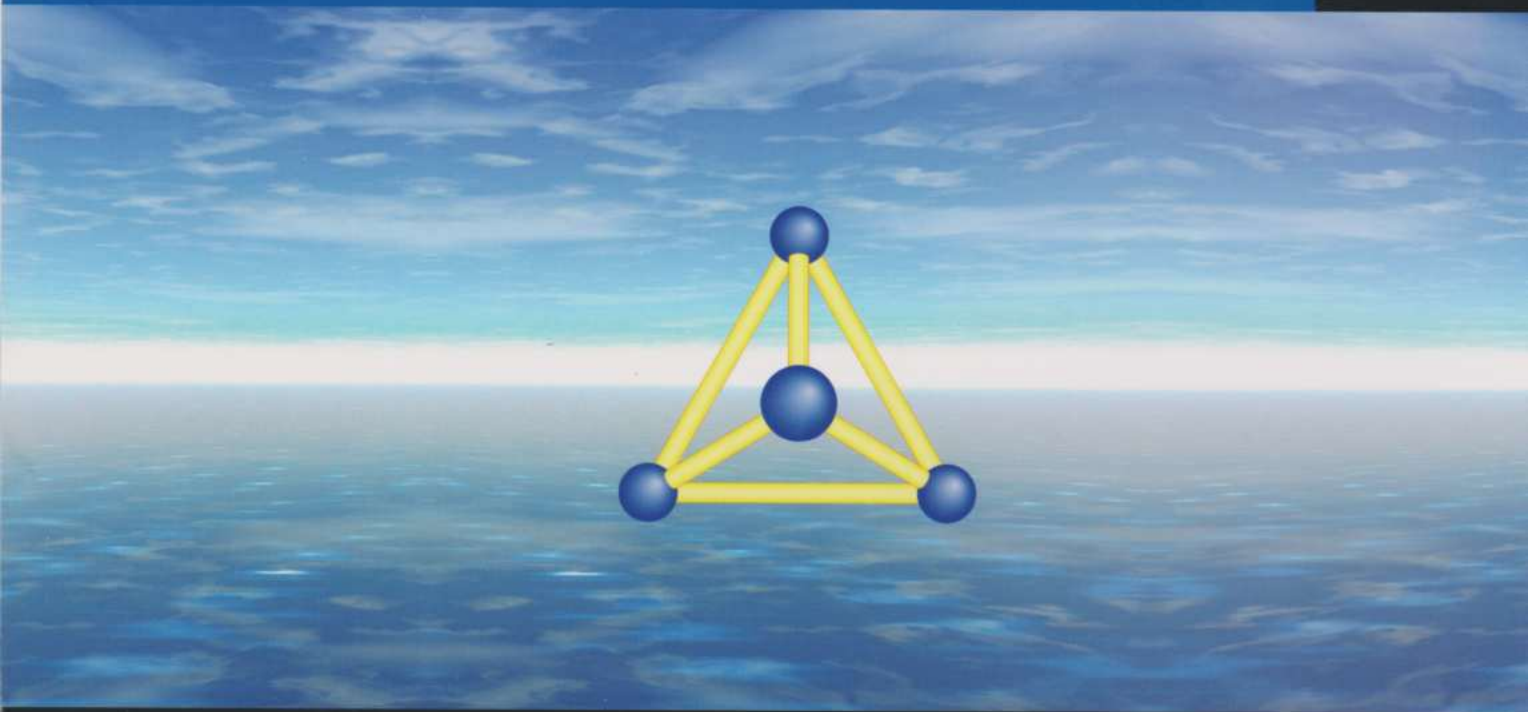
Solution 4.16

There are several possibilities — for example:



Note that $n = 9$ and $m = 24$, and so m is a multiple of $n - 1$.

Index



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